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INTERACTION OF LEGENDRE CURVES AND LAGRANGIAN SUBMANIFOLDS

BY

BANG-YEN CHEN

Department of Mathematics, Michigan State University
East Lansing, Michigan 48824-1027, USA
e-mail: bychen@math.msu.edu

ABSTRACT

It is proved in [8] that there exist no totally umbilical Lagrangian submanifolds in a complex-space-form $\widetilde{M}^n(4c)$, $n \geq 2$, except the totally geodesic ones. In this paper we introduce the notion of Lagrangian H-umbilical submanifolds which are the "simplest" Lagrangian submanifolds next to the totally geodesic ones in complex-space-forms. We show that for each Legendre curve in a 3-sphere S^3 (respectively, in a 3-dimensional anti-de Sitter space-time H_1^3), there associates a Lagrangian H-umbilical submanifold in $\mathbb{C}P^n$ (respectively, in $\mathbb{C}H^n$) via warped products. The main part of this paper is devoted to the classification of Lagrangian H-umbilical submanifolds in $\mathbb{C}P^n$ and in $\mathbb{C}H^n$. Our classification theorems imply in particular that "except some exceptional classes", Lagrangian H-umbilical submanifolds of $\mathbb{C}P^n$ and of $\mathbb{C}H^n$ are obtained from Legendre curves in S^3 or in H_1^3 via warped products. This provides us an interesting interaction of Legendre curves and Lagrangian H-umbilical submanifolds in non-flat complex-space-forms. As an immediate by-product, our results provide us many concrete examples of Lagrangian H-umbilical isometric immersions of real-space-forms into non-flat complex-space-forms.

1. Introduction

Let $f\colon M \to \widetilde{M}^n$ be an isometric immersion from a Riemannian n-manifold M into a Kaehler n-manifold $\widetilde{M}^n$. Then M is called a Lagrangian (or totally real in [7]) submanifold if the almost complex structure J of $\widetilde{M}^m$ carries each

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tangent space of M into its corresponding normal space. By a complex-space-form $\widetilde{M}^n(4c)$ we mean a Kaehler manifold with constant holomorphic sectional curvature $4c$.

An n-dimensional submanifold M of a Riemannian manifold (N, g) is called totally umbilical (respectively, totally geodesic) if its second fundamental form h in N satisfies $h(X,Y) = g(X,Y)H$ (respectively, $h \equiv 0$), where $H = \frac{1}{n}$ trace h is the mean curvature vector of M in N. For a totally umbilical submanifold M the shape operator A_H at H has exactly one eigenvalue; moreover, $A_\xi = 0$ for each normal vector ξ perpendicular to H.

Totally umbilical submanifolds, if they exist, are the simplest submanifolds next to totally geodesic submanifolds in a Riemannian manifold. However, it is proved in [8] that there exist no totally umbilical Lagrangian submanifolds in a complex-space-form $\widetilde{M}^n(4c)$ with $n \geq 2$ except the totally geodesic ones.

In view of above facts it is natural to look for and to investigate the "simplest" Lagrangian submanifolds next to the totally geodesic ones in complex-space-forms $\widetilde{M}^n(4c)$. In order to do so we introduce in this paper the concept of **Lagrangian H-umbilical submanifolds.** By a Lagrangian H-umbilical submanifold of a Kaehler manifold $\widetilde{M}^n$ we mean a Lagrangian submanifold whose second fundamental takes the following simple form:

$$(1.1)\qquad \begin{aligned} &h(e_1,e_1) = \lambda J e_1, \quad h(e_2,e_2) = \cdots = h(e_n,e_n) = \mu J e_1, \\ &h(e_1,e_j) = \mu J e_j, \quad h(e_j,e_k) = 0, \quad j \neq k, \quad j,k = 2,\ldots,n \end{aligned}$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field. It is obvious that condition (1.1) is equivalent to

$$(1.2)\qquad \begin{aligned} h(X,Y) =&\alpha \langle JX,H\rangle \langle JY,H\rangle H \\ &+ \beta \langle H,H\rangle \{\langle X,Y\rangle H + \langle JX,H\rangle JY + \langle JY,H\rangle JX\} \end{aligned}$$

for vectors X, Y tangent to M, where $\langle X,Y\rangle = g(X,Y)$ and

$$\alpha = \frac{\lambda - 3\mu}{\gamma^3}, \quad \beta = \frac{\mu}{\gamma^3}, \quad \gamma = \frac{\lambda + (n-1)\mu}{n}$$

when $H \neq 0$. Clearly, a non-minimal Lagrangian H-umbilical submanifold satisfies the following two conditions:

(a) JH is an eigenvector of the shape operator A_H and
(b) the restriction of A_H to $(JH)^\perp$ is proportional to the identity map.

On the other hand, because the second fundamental form of a Lagrangian submanifold satisfies (cf. [7])

$$\langle h(X,Y), JZ\rangle = \langle h(Y,Z), JX\rangle = \langle h(Z,X), JY\rangle \tag{1.3}$$

for vectors X, Y, Z tangent to M, Lagrangian H-umbilical submanifolds are the simplest Lagrangian submanifolds satisfying both Conditions (a) and (b). In this way we can regard Lagrangian H-umbilical submanifolds as the simplest Lagrangian submanifolds in a complex-space-form next to the totally geodesic ones.

It is proved in [4] that every Lagrangian submanifold M in a non-flat complex-space-form $\widetilde{M}^n(4c)$ satisfies the following sharp inequality:

$$|H|^2 \geq \frac{2(n+2)}{n^2(n-1)}\tau - \left(\frac{n+2}{n}\right)c, \tag{1.3}$$

where τ is the scalar curvature of M (see also [3]). It is also proved in [4] that the equality case of (1.3) holds identically if and only if M is a special Lagrangian H-umbilical submanifold; namely, it satisfies (1.1) with $\lambda = 3\mu$. From [3] we also know that the class of Lagrangian H-umbilical submanifolds includes the important class of twistor holomorphic Lagrangian surfaces in $\mathbb{C}P^2$.

The main purpose of this paper is to introduce and to classify Lagrangian H-umbilical submanifolds in non-flat complex-space-forms. In order to do so, first we observe that Legendre curves in a 3-sphere S^3 and in a 3-dimensional anti-de Sitter space-time H_1^3 are given by solutions of the following second order differential equation:

$$z''(x) = i\lambda(x)z'(x) - cz(x), \tag{1.4}$$

where $\lambda(x)$ is a real-valued function and c a nonzero constant. We then prove that for each Legendre curve in S^3 (respectively, H_1^3) there associates a Lagrangian H-umbilical submanifold of $\mathbb{C}P^n$ (respectively, $\mathbb{C}H^n$) via warped products in a natural way. The main part of this paper is devoted to the classification of Lagrangian H-umbilical submanifolds of $\mathbb{C}P^n$ and of $\mathbb{C}H^n$. Our classification theorems imply in particular that "except some exceptional cases", Lagrangian H-umbilical submanifolds of $\mathbb{C}P^n$ and of $\mathbb{C}H^n$ are obtained from Legendre curves in S^3 or in H_1^3 via warped products. This provides us an interesting interaction of

Legendre curves and Lagrangian H-umbilical submanifolds in non-flat complex-space-forms. As an immediate by-product, our results provide many concrete examples of Lagrangian H-umbilical isometric immersions of real-space-forms into complex projective spaces and complex hyperbolic spaces.

Lagrangian H-umbilical submanifolds in complex Euclidean spaces are investigated in [5]. The author introduces in [5] the notion of complex extensors, moreover, he obtains in [5] the classification of Lagrangian H-umbilical submanifolds of complex Euclidean n-space $\mathbb{C}^n$ by utilizing complex extensors of the unit hypersurface in Euclidean n-space $\mathbb{E}^n$.

2. Preliminaries

In the following, $\widetilde{M}^n(4c)$ denotes a complete simply-connected Kaehler manifold of complex dimension n with constant holomorphic sectional curvature $4c$. Let M be a Lagrangian submanifold of $\widetilde{M}^n(4c)$. We denote the Levi-Civita connections of M and of $\widetilde{M}^n(4c)$ by ∇ and $\widetilde{\nabla}$, respectively. The formulas of Gauss and Weingarten are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

for tangent vector fields X and Y and normal vector fields ξ, where D is the connection on the normal bundle. The second fundamental form h is related to A_ξ by

$$\langle h(X,Y),\xi\rangle = \langle A_\xi X, Y\rangle .$$

The mean curvature vector H of M is defined by $H = \frac{1}{n}\,\mathrm{trace}\, h$.

For Lagrangian submanifolds, we have (cf. [7])

$$D_X JY = J\nabla_X Y, \tag{2.3}$$

$$A_{JX}Y = -Jh(X,Y) = A_{JY}X. \tag{2.4}$$

The above formulas imply immediately that $\langle h(X,Y), JZ\rangle$ is totally symmetric. If we denote the curvature tensors of ∇ and D by R and R^D, respectively, i.e., $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and $R^D(X,Y) = [D_X, D_Y] - D_{[X,Y]}$, then the equations of Gauss, Codazzi and Ricci are given by

$$\langle R(X,Y)Z,W\rangle = \langle A_{h(Y,Z)}X,W\rangle - \langle A_{h(X,Z)}Y,W\rangle + c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle), \tag{2.5}$$

$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z), \tag{2.6}$$

$$\langle R^D(X,Y)JZ,JW\rangle = \langle [A_{JZ},A_{JW}]X,Y\rangle + c(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle), \tag{2.7}$$

where X, Y, Z, W (respectively, η and ξ) are vector fields tangent (respectively, normal) to M and ∇h is defined by

$$(\nabla h)(X,Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.8}$$

For a Lagrangian submanifold M in $\widetilde{M}^n(4c)$, an orthonormal frame field

$$e_1, \ldots, e_n, e_{1^*}, \ldots, e_{n^*}$$

is called an **adapted Lagrangian frame field** if $e_1, \ldots, e_n$ are orthonormal tangent vector fields and $e_{1^*}, \ldots, e_{n^*}$ are normal vector fields given by

$$e_{1^*} = Je_1, \ldots, e_{n^*} = Je_n. \tag{2.9}$$

We recall the following Existence and Uniqueness Theorems for later use (cf. [4,6]).

THEOREM A: *Let $(M^n, \langle .,. \rangle)$ be an n-dimensional simply connected Riemannian manifold. Let σ be a TM-valued symmetric bilinear form on M satisfying*

(1) *$\langle \sigma(X,Y), Z\rangle$ is totally symmetric,*
(2) *$(\nabla\sigma)(X,Y,Z) = \nabla_X\sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$ is totally symmetric,*
(3) *$R(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + \sigma(\sigma(Y,Z),X) - \sigma(\sigma(X,Z),Y)$,*

then there exists a Lagrangian isometric immersion $x\colon (M, \langle .,. \rangle) \to \widetilde{M}^n(4c)$ whose second fundamental form h is given by $h(X,Y) = J\sigma(X,Y)$.

THEOREM B: *Let $x^1, x^2\colon M \to \widetilde{M}^n(4c)$ be two Lagrangian isometric immersions of a Riemannian manifold M with second fundamental forms h^1 and h^2. If*

$$\langle h^1(X,Y), Jx^1_\star Z\rangle = \langle h^2(X,Y), Jx^2_\star Z\rangle,$$

for all vector fields X, Y, Z tangent to M, then there exists an isometry ϕ of $\widetilde{M}^n(4c)$ such that $x^1 = x^2 \circ \phi$.

Let $M = I \times_f F$ denote the warped product of an open interval I in $\mathbb{R}$ and a Riemannian manifold F with warped function f. Denote by R and R^F the Riemannian curvature tensors of M and F, respectively. Then R and R^F satisfy

$$R(X,Y)Z = R^F(X,Y)Z - \left(\frac{f'}{f}\right)^2 (\langle Y,Z\rangle X - \langle X,Z\rangle Y), \tag{2.10}$$

$$R(X,V)V = -\left(\frac{VVf}{f}\right) X, \tag{2.11}$$

for X, Y, Z tangent to F and V tangent to I. Notice that there is a difference in sign between the definition of R of this paper and that of [12].

3. A general method for constructing Lagrangian submanifolds

In this section, we mention a general method for constructing Lagrangian submanifolds both in complex projective spaces and in complex hyperbolic spaces.

CASE (1): $\widetilde{M}^n(4c) = \mathbb{C}P^n(4c)$, $c > 0$. Let

$$S^{2n+1}(c) = \{z = (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : \langle z, z\rangle = 1/c > 0\}$$

be the hypersphere of $\mathbb{C}^{n+1}$ with constant sectional curvature c centered at the origin. We consider the Hopf fibration

$$\pi \colon S^{2n+1}(c) \to \mathbb{C}P^n(4c). \tag{3.1}$$

On $S^{2n+1}(c)$ we consider the contact structure ϕ (i.e., the projection of the complex structure J of $\mathbb{C}^{n+1}$ on the tangent bundle of $S^{2n+1}(c)$) and the structure vector field $\xi = Jx$, where x is the position vector. An isometric immersion $f \colon M \to S^{2n+1}(c)$ is called C-**totally real** (or an integral submanifold) if ξ is normal to $f_*(TM)$ and $\langle \phi(f_*(TM)), f_*(TM)\rangle = 0$, where $\langle\, ,\, \rangle$ denotes the inner product on $\mathbb{C}^{n+1}$. On $\mathbb{C}^{n+1}$ we consider the complex structure J induced by $i = \sqrt{-1}$. The main results of [13] can be specialized to our situation as follows.

Let $g \colon M \to \mathbb{C}P^n(4c)$ be a Lagrangian isometric immersion. Then there exists an isometric covering map $\tau \colon \widehat{M} \to M$ and a C-totally real isometric immersion $f \colon \widehat{M} \to S^{2n+1}(c)$ such that $g(\tau) = \pi(f)$. Hence every Lagrangian immersion can be lifted locally (or globally if we assume the manifold is simply connected) to a C-totally real immersion of the same Riemannian manifold. Conversely, let $f\colon \widehat{M} \to S^{2n+1}(c)$ be a C-totally real isometric immersion. Then $g = \pi(f)\colon M \to \mathbb{C}P^n(4c)$ is again an isometric immersion, which is Lagrangian. Under this correspondence, the second fundamental forms h^f and h^g of f and g satisfy $\pi_* h^f = h^g$. Moreover, h^f is horizontal with respect to π. (We shall denote h^f and h^g simply by h.)

CASE (2): $\widetilde{M}^n(4c) = \mathbb{C}H^n(c)$, $c < 0$. In this case, we consider the complex number $(n+1)$-space $\mathbb{C}_1^{n+1}$ endowed with the pseudo-Euclidean metric g_0 given by

$$g_0 = -dz_1 d\bar{z}_1 + \sum_{j=2}^{n+1} dz_j d\bar{z}_j. \tag{3.2}$$

Put

$$H_1^{2n+1}(c) = \{z = (z_1, z_2, \dots, z_{n+1})\colon \langle z, z\rangle = 1/c < 0\}, \tag{3.3}$$

where $\langle\, ,\rangle$ denotes the inner product on $\mathbb{C}_1^{n+1}$ induced from g_0. $H_1^{2n+1}(c)$ is known as an anti-de Sitter space-time.

We put

$$T'_z = \{z \in \mathbb{C}^{n+1}\colon \mathrm{Re}\,\langle u, z\rangle = \mathrm{Re}\,\langle u, iz\rangle = 0\}, \quad H_1^1 = \{\lambda \in \mathbb{C}\colon \lambda\bar{\lambda} = 1\}.$$

Then we have an H_1^1-action on $H_1^{2n+1}(c)$, $z \mapsto \lambda z$ and at each point $z \in H_1^{2n+1}(c)$, the vector iz is tangent to the flow of the action. Since the metric g_0 is Hermitian, we have $\mathrm{Re}\, g_0(iz, iz) = 1/c$. Note that the orbit is given by $x_t = (\cos t + i \sin t)z$ and $dx_t/dt = iz_t$. Thus the orbit lies in the negative definite plane spanned by z and iz. The quotient space $H_1^{2n+1}/\sim$, under the identification induced from the action, is the complex hyperbolic space $\mathbb{C}H^n(4c)$ with constant holomorphic sectional curvature $4c$, with the complex structure J induced from the canonical complex structure J on $\mathbb{C}_1^{n+1}$ via the following totally geodesic fibration:

$$\pi\colon H_1^{2n+1}(c) \to \mathbb{C}H^n(4c). \tag{3.4}$$

Just as in Case (1), let $g\colon M \to \mathbb{C}H^n(4c)$ be a Lagrangian isometric immersion. Then there exists an isometric covering map $\tau\colon \widehat{M} \to M$ and a C-totally real isometric immersion $f\colon \widehat{M} \to H_1^{2n+1}(c)$ such that $g(\tau) = \pi(f)$. Hence every totally real immersion can be lifted locally (or globally if we assume the manifold is simply connected) to a C-totally real immersion. Conversely, let $f\colon \widehat{M} \to H_1^{2n+1}(c)$ be a C-totally real isometric immersion. Then $g = \pi(f)\colon M \to \mathbb{C}H^n(4c)$ is again an isometric immersion, which is Lagrangian. Similarly, under this correspondence, the second fundamental forms h^f and h^g of f and g satisfy $\pi_* h^f = h^g$. Moreover, h^f is horizontal with respect to π. (We shall also denote h^f and h^g simply by h.)

C-totally real curves in $S^3(c)$ (or in $H_1^3(c)$) are known as **Legendre curves.**

4. Legendre curves and warped Lagrangian H-umbilical submanifolds

In this section we prove that Legendre curves in $S^3(c)$ and $H_1^3(c)$ are given by solutions of the second order differential equation $z''(x) = i\lambda(x)z'(x) - cz(x)$. Moreover, for each such Legendre curve, we construct its corresponding canonical Lagrangian H-umbilical submanifolds of $\mathbb{C}P^n(4c)$.

THEOREM 4.1: *Let c be a positive number and $z = (z_1, z_2)\colon I \to S^3(c) \subset \mathbb{C}^2$ be a unit speed curve where I is either an open interval or a circle. Then the following statements hold.*

(a) *If $z\colon I \to \mathbb{C}^2$ satisfies differential equation*

$$z''(x) = i\lambda(x)z'(x) - cz(x) \tag{4.1}$$

for some nonzero real-valued function λ on I, it defines a Legendre curve in $S^3(c)$. Conversely, if z defines a Legendre curve in $S^3(c)$, it satisfies differential equation (4.1) for some real-valued function λ.

(b) *If $z\colon I \to S^3(c) \subset \mathbb{C}^2$ is a unit speed Legendre curve such that $|z_2(x)|$ is a positive function on I, then*

(b-1) *the map $\psi\colon I \times S^{n-1}(1) \to \mathbb{C}^{n+1}$, given by*

$$\psi(x, y_1, \dots, y_n) = (z_1(x), z_2(x)y_1, \dots, z_2(x)y_n) \tag{4.2}$$

with $y_1^2 + \cdots + y_n^2 = 1$, defines a C-totally real isometric immersion

$$\psi\colon I \times_{|z_2|} S^{n-1}(1) \to S^{2n+1}(c). \tag{4.3}$$

(b-2) *The map (4.2) gives rise to a Lagrangian H-umbilical isometric immersion*

$$\breve{\psi} = \pi \circ \psi\colon I \times_{|z_2|} S^{n-1}(1) \to \mathbb{C}P^n(4c), \tag{4.4}$$

such that, with respect to some orthonormal local frame field $e_1, \dots, e_n$ with $e_1 = \partial/\partial x$, the second fundamental form h of $\bar\psi$ satisfies

$$h(e_1,e_1) = \lambda J e_1, \quad h(e_2,e_2) = \dots = h(e_n,e_n) = \mu J e_1, \\ h(e_1,e_j) = \mu J e_j, \quad h(e_j,e_k) = 0, \quad j \neq k, \quad j,k = 2,\dots,n, \tag{4.5}$$

where $\mu = -\frac{1}{|z_2|^2}\langle z_2, iz_2'\rangle$ and $\langle z_2, iz_2'\rangle$ denotes the real part of $z_2\bar z_2'$.

(c) *For a unit speed Legendre curve $z\colon I \to S^3(c) \subset \mathbb{C}^2$ we have*

(c-1) *the map $\phi\colon S^{n-1}(c) \times I \to \mathbb{C}^{n+1}$, given by*

$$\phi(x, y_1, \dots, y_n) = (z_1(x)y_1, z_2(x)y_1, \frac{1}{\sqrt{c}}y_2, \dots, \frac{1}{\sqrt{c}}y_n) \tag{4.6}$$

with $y_1^2 + \dots + y_n^2 = 1$, defines a C-totally real isometric immersion from the warped product manifold $S^{n-1}(c) \times_{y_1} I$ into $S^{2n+1}(c)$,

(c-2) *the warped product $N^n(c) = S^{n-1}(c) \times_{y_1} I$ is of constant sectional curvature c and the immersion $\bar\phi = \pi \circ \phi$ gives rise to a Lagrangian H-umbilical isometric immersion from $N^n(c)$ into $\mathbb{C}P^n(4c)$,*

(c-3) *if the Legendre curve γ is a closed curve, then $N^n(c)$ is compact and the universal lift of the Lagrangian immersion $\bar\phi = \pi \circ \phi\colon N^n(c) \to \mathbb{C}P^n(4c)$ is a Lagrangian H-umbilical isometric immersion of $S^n(c)$ into $\mathbb{C}P^n(4c)$,*

(c-4) *the second fundamental form h of the Lagrangian immersion $\pi \circ \phi$ takes the following form:*

$$h(e_1,e_1) = \frac{\lambda}{y_1} J e_1, \quad h(e_1,e_j) = 0, \quad h(e_j,e_k) = 0, \quad j,k = 2,\dots,n, \tag{4.7}$$

with respect to some suitable orthonormal local frame fields.

Proof: (a) Suppose $z\colon I \to S^3(c) \subset \mathbb{C}^2$ is a unit speed curve which satisfies (4.1). Then, by taking the derivative of $\langle iz', iz\rangle = 0$ and applying $\langle z, z\rangle = 1/c$, $\langle z', z'\rangle = 1$ and equation (4.1), we have $\langle z', iz\rangle\lambda = 0$. Let $U = \{x \in I\colon \lambda(x) \neq 0\}$. If $U = I$, then $\langle z', iz\rangle = 0$ identically on I. Thus, z defines a Legendre curve in $S^3(c)$. Now, suppose $U \neq I$, then we have $\lambda = 0$ on the open subset $I - U$. Since $\langle z', iz\rangle' = 0$ on $I - U$, the continuity implies $\langle z', iz\rangle = 0$ identically. Therefore, z defines a Legendre curve in $S^3(c)$.

Conversely, if z defines a Legendre curve in $S^3(c)$, then $\langle z', iz\rangle = 0$ identically. Thus, we have $\langle z'', iz\rangle = 0$. Since z, iz, z' and iz' form an orthogonal frame field along the Legendre curve, $z''(x) = i\lambda(x)z'(x) + k(x)z(x)$ for some real-valued functions λ and k. On the other hand, from $\langle z,z\rangle = 1/c$ and $\langle z',z'\rangle = 1$, we also have $\langle z'',z\rangle = -1$. Thus, $k(x) = -c$ which implies (4.1).

(b) It is easy to verify that map (4.2) defines an immersion from $I \times S^{n-1}(1)$ into $\mathbb{C}^{n+1}$ whose induced metric is the warped product metric $g = ds^2+|z_2(s)|^2 g_0$, where g_0 is the standard metric on the unit $(n-1)$-sphere $S^{n-1}(1)$.

Since z is assumed to define a Legendre curve in $S^3(c)$, we have $\langle z,z\rangle = 1/c$. By combining this with the condition $y_1^2 + \cdots + y_n^2 = 1$, we get $\langle \psi,\psi\rangle = 1/c$. Thus $I \times_{|z_2|} S^{n-1}(1)$ is mapped into $S^{2n+1}(c)$.

Since $\langle z', iz\rangle = 0$ by hypothesis, (4.2) yields $\langle \psi_x, i\psi\rangle = 0$. For each fixed $x \in I$, (4.2) implies that $\{x\} \times S^{n-1}(1)$ is immersed as a C-totally real submanifold of $S^{2n+1}(c)$ in a natural way. Hence, $\langle X, i\phi\rangle = 0$ for any X tangent to the second component of the warped product. Therefore, $i\psi$ is normal to the warped product $I \times_{|z_2|} S^{n-1}(1)$.

By using (4.2), it is easy to verify that the contact structure ϕ on $S^{2n+1}(c)$ maps each tangent vector of the warped product into a normal vector. Consequently, ψ is a C-totally real isometric immersion and from the discussion given in Case (1) in section 3, we conclude that the composition $\bar{\psi} = \pi \circ \psi$ is indeed a Lagrangian isometric immersion from $I \times_{|z_2|} S^{n-1}(1)$ into $\mathbb{C}P^n(4c)$. By a direct straight-forward computation, we obtain (4.5). This proves Statement (b). Statement (c) follow from a straight-forward long computation, too. ∎

Similarly, we have the following.

THEOREM 4.2: *Let c be a negative number and $z = (z_1, z_2)\colon I \to H_1^3(c) \subset \mathbb{C}_1^2$ be a unit speed curve where I is an open interval.*

(1) *If $z\colon I \to \mathbb{C}_1^2$ satisfies differential equation*

$$z''(x) = i\lambda z'(x) - cz \tag{4.8}$$

for some nonzero real-valued function λ on I, then it defines a Legendre curve in $H_1^3(c)$. Conversely, if z defines a Legendre curve in $H_1^3(c)$, then it satisfies differential equation (4.8) for some real-valued function λ.

(2) *If $z\colon I \to H_1^3(c) \subset \mathbb{C}_1^2$ is a unit speed Legendre curve in $H_1^3(c)$ such that $|z_1|$ is a positive function, then the map $\phi_1\colon I \times H^{n-1}(-1) \to \mathbb{C}_1^{n+1}$ given by*

$$\psi_1(x, y_1, \ldots, y_n) = (z_1(x)y_1, \ldots, z_1(x)y_n, z_2(x)) \tag{4.9}$$

with $y_1^2 - y_2^2 - \cdots - y_n^2 = 1$ *defines a C-totally real isometric immersion*

$$\psi_1 \colon I \times_{|z_1|} H^{n-1}(-1) \to H_1^{2n+1}(c). \tag{4.10}$$

(4.9) *gives rise to a Lagrangian H-umbilical isometric immersion*

$$\bar{\psi}_1 = \pi \circ \psi_1 \colon I \times_{|z_1|} H^{n-1}(-1) \to \mathbb{C}H^n(4c). \tag{4.11}$$

(3) *With respect to some orthonormal local frame fields* $e_1, \ldots, e_n$ *with* $e_1 = \partial/\partial x$, *the second fundamental form* h *of* $\bar{\psi}_1$ *satisfies*

$$\begin{aligned} &h(e_1,e_1) = \lambda J e_1, \quad h(e_j,e_j) = \mu J e_1, \quad \mu = -\frac{1}{|z_1|^2}\langle z_1, i z_1' \rangle, \\ &h(e_1,e_j) = \mu J e_j, \quad h(e_j,e_k) = 0, \quad j \neq k, \quad j,k = 2,\ldots,n. \end{aligned} \tag{4.12}$$

(4) *If* $z \colon I \to H_1^3(c) \subset \mathbb{C}_1^2$ *is a unit speed Legendre curve in* $H_1^3(c)$ *such that* $|z_2|$ *is a positive function, then the map* $\phi_2 \colon I \times S^{n-1}(1) \to \mathbb{C}_1^{n+1}$ *given by*

$$\psi(x, y_1, \ldots, y_n) = (z_1(x), z_2(x)y_1, \ldots, z_2(x)y_n) \tag{4.13}$$

with $y_1^2 + y_2^2 + \cdots + y_n^2 = 1$ *defines a C-totally real isometric immersion*

$$\psi_2 \colon I \times_{|z_2|} S^{n-1}(1) \to H_1^{2n+1}(c). \tag{4.14}$$

(4.14) *gives rise to a Lagrangian H-umbilical isometric immersion*

$$\bar{\psi}_2 = \pi \circ \psi_2 \colon I \times_{|z_2|} S^{n-1}(1) \to \mathbb{C}H^n(4c) \tag{4.15}$$

such that, with respect to some orthonormal local frame field $e_1, \ldots, e_n$ *with* $e_1 = \partial/\partial x$, *the second fundamental form* h *of* $\bar{\psi}_1$ *satisfies*

$$\begin{aligned} &h(e_1,e_1) = \lambda J e_1, \quad h(e_j,e_j) = \mu J e_1, \quad \mu = -\frac{1}{|z_2|^2}\langle z_2, i z_2' \rangle, \\ &h(e_1,e_j) = \mu J e_j, \quad h(e_j,e_k) = 0, \quad j \neq k, \quad j,k = 2,\ldots,n. \end{aligned} \tag{4.16}$$

Because the proof of this theorem is similar to the proof of Theorem 4.1, we omit it.

According to Theorems 4.1 and 4.2, one can construct many Lagrangian H-umbilical submanifolds in $\mathbb{C}P^n(4c)$ and in $\mathbb{C}H^n(4c)$ by using unit speed curves in S^3 and in H_1^3 which satisfy differential equations (4.1) and (4.8), respectively.

Remark 4.3: Unit speed curves in $S^3(c)$ (respectively, in $H_1^3(c)$) which satisfy the differential equation $z''(s) = -cz(s)$ are geodesics and they are not necessarily Legendre curves.

5. Lagrangian H-umbilical immersions of real-space-forms into $\mathbb{C}P^n$

From statement (c) of Theorem 4.1 we know that real-space-forms admit many Lagrangian H-umbilical isometric immersions into $\mathbb{C}P^n$. In this section, we study Lagrangian H-umbilical isometric immersions of a real-space-form $M^n(\delta^2)$ into $\mathbb{C}P^n(4c)$ for $\delta^2 > c$.

THEOREM 5.1: *Let c be a positive number and $n \geq 2$. Then*

(i) *A simply-connected open portion M of the Riemannian n-sphere $S^n(\delta^2)$ with curvature $\delta^2 > c$ admits a Lagrangian H-umbilical isometric immersion ℓ: $M \to \mathbb{C}P^n(4c)$ such that*

$$\begin{aligned} &h(e_1,e_1) = 2bJe_1, \; h(e_2,e_2) = \cdots = h(e_n,e_n) = bJe_1, \\ &h(e_1,e_j) = bJe_j, \quad h(e_j,e_k) = 0, \quad j \neq k, \quad j,k = 2,\dots,n, \end{aligned} \tag{5.1}$$

for some suitable orthonormal local frame field $e_1,\dots,e_n$ on M, where b is the constant given by $b = \sqrt{\delta^2 - c}$.

(ii) *Let L: $M \to \mathbb{C}P^n(4c)$ be a Lagrangian H-umbilical isometric immersion satisfying* (5.1) *for some non-trivial function b, then*

(ii-1) *b is constant,*

(ii-2) *M is an open portion of $S^n(\delta^2)$ with $\delta^2 = b^2 + c$ and hence M is locally isometric to the warped product $I \times_{\frac{1}{\delta}\cos(\delta x)} S^{n-1}(1)$,*

(ii-3) *up to rigid motions of $\mathbb{C}P^n(4c)$, the immersion L is given by the immersion ℓ mentioned in Statement* (i), *and moreover*

(ii-4) *L is the composition $\pi \circ \phi$, where π is the projection of Hopf's fibration and ϕ: $\hat{M} \to S^{2n+1}(c)$ $(\subset \mathbb{C}^{n+1})$ is given by*

(5.2)
$$\begin{aligned} \phi(x,y_1,\dots,y_n) = \frac{e^{ibx}}{2\delta^2}\Bigg(&\Big(\frac{b(b-\delta)}{\sqrt{c}} + \sqrt{c}\,y_1\Big)e^{i\delta x} + \Big(\frac{b(b+\delta)}{\sqrt{c}} + \sqrt{c}\,y_1\Big)e^{-i\delta x}, \\ &(\delta - b + by_1)e^{i\delta x} - (\delta + b - by_1)e^{-i\delta x}, \\ &\delta y_2(e^{i\delta x} + e^{-i\delta x}),\dots,\delta y_n(e^{i\delta x} + e^{-i\delta x})\Bigg), \\ &y_1^2 + \cdots + y_n^2 = 1, \quad b = \sqrt{\delta^2 - c}, \end{aligned}$$

and $\hat{M}$ is the covering space of M via the Hopf fibration.

Proof: Let M be a simply-connected open portion of $S^n(\delta^2)$. Then M is isometric to an open subset of the warped product $I \times_{\frac{1}{\delta}\cos(\delta x)} S^{n-1}(1)$ with warped metric

$$g = dx^2 + \frac{1}{\delta^2}\cos^2(\delta x) g_0, \tag{5.3}$$

where g_0 is the standard metric on $S^{n-1}(1)$. With respect to a spherical coordinate system $\{u_2, \dots, u_n\}$ on $S^{n-1}(1)$, we have

$$g_0 = du_2^2 + \cos^2 u_2 du_3^2 + \cdots + \cos^2 u_2 \cdots \cos^2 u_{n-1} du_n^2. \tag{5.4}$$

From (5.3) and (5.4) we get

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial u_k} = -\delta \tan(\delta x) \frac{\partial}{\partial u_k}, \quad \nabla_{\frac{\partial}{\partial u_2}} \frac{\partial}{\partial u_2} = \frac{\sin(2\delta x)}{2\delta} \frac{\partial}{\partial x}, \\
&\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = -\tan u_i \frac{\partial}{\partial u_j}, \quad 2 \le i < j, \\
&\nabla_{\frac{\partial}{\partial u_j}} \frac{\partial}{\partial u_j} = \frac{\sin(2\delta x)}{2\delta} \prod_{\ell=2}^{j-1} \cos^2 u_\ell \frac{\partial}{\partial x} + \sum_{k=2}^{j-1} \left(\frac{\sin 2u_k}{2} \prod_{\ell=k+1}^{j-1} \cos^2 u_\ell \right) \frac{\partial}{\partial u_k},
\end{aligned} \tag{5.5}$$

where $2 \le i, j, k \le n$.

Define a TM-valued symmetric bilinear form σ on M by

$$\begin{aligned}
&\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = 2b\frac{\partial}{\partial x}, \quad \sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u_i}\right) = b\frac{\partial}{\partial u_i}, \\
&\sigma\left(\frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k}\right) = b\delta_{jk}\delta^{-2}\cos^2(\delta x) \prod_{\ell=2}^{j-1} \cos^2 u_\ell \frac{\partial}{\partial x}, \quad j, k = 2, \dots, n,
\end{aligned} \tag{5.6}$$

where $b = \sqrt{\delta^2 - c}$ and δ_{jk} denotes the Kronecker deltas.

By applying (5.5) and (5.6) and by a long computation, we may prove that M together with the symmetric bilinear form σ satisfies the three conditions mentioned in the Existence Theorem (Theorem A). Therefore, there exists a Lagrangian H-umbilical isometric immersion ℓ: $M \to \mathbb{C}P^n(4c)$ with $h = J\sigma$ as its second fundamental form. This proves Statement (i).

For Statement (ii), let us assume that L: $M \to \mathbb{C}P^n(4c)$ is a Lagrangian H-umbilical isometric immersion whose second fundamental form satisfies

$$\begin{aligned}
&h(e_1, e_1) = 2bJe_1, h(e_2, e_2) = \cdots = h(e_n, e_n) = bJe_1, \\
&h(e_1, e_j) = bJe_j, \quad h(e_j, e_k) = 0, \quad j \ne k, \quad j, k = 2, \dots, n,
\end{aligned} \tag{5.7}$$

for some non-trivial function b. Then we have

$$\begin{aligned}\bar{\nabla}_{e_1}h(e_j,e_1)&=D_{e_1}h(e_j,e_1)-h(\nabla_{e_1}e_j,e_1)-h(e_j,\nabla_{e_1}e_1)=(e_1b)Je_j,\\ \bar{\nabla}_{e_j}h(e_1,e_1)&=2(e_jb)Je_1,\quad j=2,\dots,n.\end{aligned}\tag{5.8}$$

Using (5.8) and Codazzi's equation (2.6) we conclude that b is a constant. Therefore, by applying (5.7) and Gauss' equation (2.5), we know that M is of constant sectional curvature $\delta^2=b^2+c$. Consequently, the Uniqueness Theorem (Theorem B) implies that up to rigid motions of $\mathbb{C}P^n(4c)$, $\ell=L$.

Since $L\colon M\to\mathbb{C}P^n(4c)$ is a Lagrangian H-umbilical isometric immersion satisfying (5.1) for some constant $b\neq 0$, M is of constant sectional curvature $\delta^2=b^2+c$. Hence M is an open portion of $S^n(\delta^2)$ which is locally isometric to an open portion of the warped product $I\times_{\frac{1}{\delta}\cos(\delta x)}S^{n-1}(1)$ with $I=(-\frac{\pi}{2\delta},\frac{\pi}{2\delta})$. So, we may assume the metric of M (and hence of $\hat{M}$) is given by (5.3)–(5.4).

Let $\phi\colon\hat{M}\to S^{2n+1}(c)\subset\mathbb{C}^{n+1}$ be a horizontal lift of the Lagrangian immersion $L\colon M\to\mathbb{C}P^n(4c)$ via Hopf's fibration. Then, by (5.1), (5.5), (5.6) and Gauss' formula (2.1), we have

$$\phi_{xx}=2bi\,\phi_x-c\phi,\quad \phi_x=\frac{\partial\phi}{\partial x},\quad \phi_{xx}=\frac{\partial^2\phi}{\partial x^2},\tag{5.9}$$

$$\tilde{\nabla}_Y\phi_x=(ib-\delta\tan(\delta x))Y,\tag{5.10}$$

$$\tilde{\nabla}_Y\tilde{\nabla}_Z\phi=\langle Y,Z\rangle\{(ib)\phi_x-c\phi\}+\nabla_YZ,\tag{5.11}$$

where Y,Z are vector fields tangent to the second component $S^{n-1}(1)$ of the warped product and ∇_YZ is the tangential component of $\tilde{\nabla}_YZ$.

Let $\{u_2,\dots,u_n\}$ be a spherical coordinate system on $S^{n-1}(1)$. By solving (5.9) we obtain

$$\phi(x,u_2,\dots,u_n)=A(u_2,\dots,u_n)e^{(b+\delta)ix}+B(u_2,\dots,u_n)e^{(b-\delta)ix},\tag{5.12}$$

for some $\mathbb{C}^{n+1}$-valued vector functions A,B. From (5.10) and (5.12), we find

$$\frac{\partial A}{\partial u_j}=\frac{\partial B}{\partial u_j},\quad j=2,\dots,n.\tag{5.13}$$

Using (5.12) and $\delta^2=b^2+c$, we get

$$ib\frac{\partial\phi}{\partial x}-c\phi=-\delta\left((b+\delta)Ae^{(b+\delta)ix}-(b-\delta)Be^{(b-\delta)ix}\right).\tag{5.14}$$

From (5.12) we have

$$\frac{\partial^2\phi}{\partial {u_2}^2} = \frac{\partial^2 A}{\partial {u_2}^2}e^{(b+\delta)ix} + \frac{\partial^2 B}{\partial {u_2}^2}e^{(b-\delta)ix}. \tag{5.15}$$

On the other hand, (5.3)–(5.5), (5.10) and (5.14) yield

$$\frac{\partial^2\phi}{\partial {u_2}^2} = -\frac{1}{2\delta}((b+\delta)A-(b-\delta)B)(e^{(b+\delta)ix}+e^{(b-\delta)ix}). \tag{5.16}$$

Combining (5.15) and (5.16), we find

$$\frac{\partial^2 A}{\partial {u_2}^2} = \frac{\partial^2 B}{\partial {u_2}^2} = -\frac{1}{2\delta}((b+\delta)A-(b-\delta)B). \tag{5.17}$$

(5.13) and (5.17) imply

$$\frac{\partial^3 A}{\partial {u_2}^3} + \frac{\partial A}{\partial u_2} = 0, \quad \frac{\partial^3 B}{\partial {u_2}^3} + \frac{\partial B}{\partial u_2} = 0. \tag{5.18}$$

From (5.17) and (5.18) we obtain

$$\begin{aligned} A &= b_0 + b_1\sin u_2 + b_2\cos u_2, \\ B &= \left(\frac{b+\delta}{b-\delta}\right) b_0 + b_1\sin u_2 + b_2\cos u_2, \end{aligned} \tag{5.19}$$

where b_0, b_1, b_2 are $\mathbb{C}^{n+1}$-valued functions of $u_3, \dots, u_n$.

If $n=2$, b_0, b_1, b_2 are constant vectors in $\mathbb{C}^3$. Thus (5.12) and (5.19) imply

(5.20)

$$\phi(x,u) = e^{bix}\left\{b_0\left(e^{\delta ix} + \left(\frac{b+\delta}{b-\delta}\right)e^{-\delta ix}\right) + (b_1\sin u + b_2\cos u)(e^{\delta ix}+e^{-\delta ix})\right\},$$

where $u=u_2$. If we choose the following initial conditions:

$$\phi(0,0) = (1/\sqrt{c},0,0), \quad \phi_x(0,0) = (0,i,0), \quad \phi_u(0,0) = (0,0,1/2\delta),$$

then (5.20) implies

(5.21)

$$\begin{aligned} \phi(x,u) = \frac{e^{bix}}{2\delta^2}\Bigg(&\left(\frac{b(b-\delta)}{\sqrt{c}} + \sqrt{c}\cos u\right)e^{\delta ix} + \left(\frac{b(b+\delta)}{\sqrt{c}} + \sqrt{c}\cos u\right)e^{-\delta ix}, \\ &(\delta - b + b\cos u)e^{\delta ix} - (\delta + b - b\cos u)e^{-\delta ix}, \delta\sin u(e^{\delta ix}+e^{-\delta ix})\Bigg). \end{aligned}$$

If $n > 2$, then (5.5), (5.12), (5.19) and (5.11) with $Y = \partial/\partial u_2, Z = \partial/\partial u_3$ imply $\partial b_1/\partial u_3 = \partial b_2/\partial u_3 = 0$. Thus

$$b_0 = b_0(u_4, \dots, u_n), \quad b_1 = b_1(u_4, \dots, u_n). \tag{5.22}$$

Similarly, (5.5), (5.12), (5.19), (5.22) and (6.11) with $Y = Z = \partial/\partial u_3$ imply

$$b_2 = b_3(u_4, \dots, u_n) \sin u_3 + b_4(u_4, \dots, u_n) \cos u_3. \tag{5.23}$$

By continuing this procedure $(n-1)$ times, we will obtain

(5.24)
$$\begin{aligned}\phi =& e^{bix}\{c_0 \left(e^{\delta ix} + (\frac{b+\delta}{b-\delta}) e^{-\delta ix}\right) \\ &+ (e^{\delta ix} + e^{-\delta ix})(c_1 \sin u_2 + c_2 \cos u_2 \sin u_3 + \cdots + c_n \cos u_2 \cdots \cos u_n)\}\end{aligned}$$

for some constant vectors $c_0, c_1, \dots, c_n \in \mathbb{C}^{n+1}$. By choosing the following initial conditions:

(5.25)
$$\begin{aligned}&\phi(0, \dots, 0) = (1/\sqrt{c}, 0, \dots, 0), \quad \frac{\partial \phi}{\partial x}(0, \dots, 0) = (0, i, 0, \dots, 0), \\ &\partial\phi/\partial u_2(0, \dots, 0) = (0, \dots, 0, 1/2\delta), \dots, \frac{\partial \phi}{\partial u_n}(0, \dots, 0) = (0, 0, 1/2\delta, 0, \dots, 0),\end{aligned}$$

we obtain (5.2), with

$$y_1 = \prod_{j=2}^{n} \cos u_j, \quad y_2 = \sin u_n \prod_{j=2}^{n-1} \cos u_j, \quad \dots, \quad y_n = \sin u_2.$$

This proves Statement (iii). ∎

6. Lagrangian H-umbilical immersions of real-space-forms into $\mathbb{C}H^n$

In this section we establish the following Existence and Classification Theorem for Lagrangian H-umbilical isometric immersions of real-space-forms into complex hyperbolic spaces.

THEOREM 6.1: *Let c be a negative number and $n \geq 2$. Then we have*

(i) *Every real-space-form $M^n(\hat{c})$ of constant sectional curvature $\hat{c}$ with $\hat{c} > c$ admits (at least locally) a Lagrangian H-umbilical isometric immersion $\ell \colon M^n(\hat{c}) \to \mathbb{C}H^n(4c)$ whose second fundamental form satisfies*

$$\begin{aligned}&h(e_1, e_1) = 2bJe_1, h(e_2, e_2) = \cdots = h(e_n, e_n) = bJe_1, \\ &h(e_1, e_j) = bJe_j, \quad h(e_j, e_k) = 0, \quad j \neq k, \quad j, k = 2, \dots, n,\end{aligned} \tag{6.1}$$

with respect to some suitable orthonormal local frame fields $e_1, \dots, e_n$ *on* $M^n(\hat{c})$, *where* b *is the constant given by* $b = \sqrt{\hat{c} - c}$.

(ii) *Let* $L\colon M \to \mathbb{C}H^n(4c)$ *be a Lagrangian* H*-umbilical isometric immersion satisfying* (6.1) *for some non-trivial function* b, *then*

(ii-1) b *is constant,*

(ii-2) M *is a real-space-form* $M^n(\hat{c})$ *of constant sectional curvature* $\hat{c} = b^2 + c$, *and*

(ii-3) *up to rigid motions of* $\mathbb{C}H^n(4c)$, L *is the immersion* ℓ *given in Statement* (i).

(iii) *Let* $L\colon M \to \mathbb{C}H^n(4c)$ *be a Lagrangian* H*-umbilical isometric immersion satisfying* (6.1). *Then* M *is locally isometric to one of the following warped products:*

$$I \times_{\frac{1}{\delta}\cos(\delta x)} S^{n-1}(1), \mathbb{R} \times_1 E^{n-1}, \mathbb{R} \times_{e^{\delta x}} E^{n-1}, \quad \text{when } n \geq 3,$$
$$I \times_{\frac{1}{\delta}\cos(\delta x)} \mathbb{R}, \quad \mathbb{R} \times_1 \mathbb{R}, \quad \mathbb{R} \times_{e^{\delta x}} \mathbb{R}, \quad \text{when } n = 2;$$

and up to rigid motions of $\mathbb{C}H^n(4c)$, L *is the composition* $\pi \circ \phi$, *where* π *is the projection from* $H_1^{2n+1}(c)$ *onto* $\mathbb{C}H^n(4c)$ *mentioned in section 3,* $\delta = \sqrt{|\hat{c}|}$, *and*

(iii-1) *when* $\hat{c} = b^2 + c > 0$, $\phi\colon M \to H_1^{2n+1}(c) \subset \mathbb{C}_1^{n+1}$ *is given by*

$$\begin{aligned} \phi(x, y_1, \dots, y_n) = \frac{e^{ibx}}{2\delta^2}\Bigg(&(\frac{b(b-\delta)}{\sqrt{-c}} - \sqrt{-c}\, y_1)e^{i\delta x} \\ &+ (\frac{b(b+\delta)}{\sqrt{-c}} - \sqrt{-c}\, y_1)e^{-i\delta x}, (\delta - b + by_1)e^{i\delta x} - (\delta + b - by_1)e^{-i\delta x}, \\ &\delta y_2(e^{i\delta x} + e^{-i\delta x}), \dots, \delta y_n(e^{i\delta x} + e^{-i\delta x})\Bigg), \end{aligned} \tag{6.2-1}$$

where $y_1^2 + \cdots + y_n^2 = 1$ *and* $\delta = \sqrt{\hat{c}}$;

(iii-2) *when* $\hat{c} = b^2 + c = 0$, $\phi\colon M \to H_1^{2n+1}(c) \subset \mathbb{C}_1^{n+1}$ *is given by*

$$\begin{aligned} \phi(x, y_1, \dots, y_n) = \frac{e^{i\sqrt{-c}x}}{2\delta^2}\Bigg(&\frac{1}{\sqrt{-c}} - ix + \frac{\sqrt{-c}}{2}\sum_{j=2}^{n} u_j^2, \\ &x + \frac{i}{2}\sum_{j=2}^{m} u_j^2, u_2, \dots, u_n\Bigg); \end{aligned} \tag{6.2-2}$$

(iii-3) *when* $\hat{c} = b^2 + c < 0$, $\phi\colon M \to H_1^{2n+1}(c) \subset \mathbf{C}_1^{n+1}$ *is given by*
(6.2-3)

$$\phi(x, u_2, \dots, u_n) = \frac{e^{ibx}}{2}\left(\frac{1}{\sqrt{-c}}(e^{\delta x}\left(1 - \frac{b}{\delta}i - c\sum_{j=2}^{n} u_j^2\right) + e^{-\delta x}(1 + \frac{b}{\delta}i)),\right.$$
$$\left. e^{\delta x}(\frac{1}{\delta} + (bi - \delta)\sum_{j=2}^{n} u_j^2) - \frac{1}{\delta}e^{-\delta x}, 2u_2e^{\delta x}, \dots, 2u_ne^{\delta x}\right), \quad \delta = \sqrt{-\hat{c}}.$$

Proof: We may assume that $M^n(\hat{c})$ is simply-connected, by taking the universal covering of $M^n(\hat{c})$ if necessary. Since $M^n(\hat{c})$ is of constant sectional curvature, $M^n(\hat{c})$ is locally isometric to an open portion of the warped product $I \times_{\frac{1}{\delta}\cos(\delta x)} S^{n-1}(1)$ (respectively, $I \times_1 E^{n-1}$ or $\mathbf{R} \times_{e^{\delta x}} E^{n-1}$) if $\hat{c} = \delta^2 > 0$ (respectively, $\hat{c} = 0$ or $\hat{c} = -\delta^2 < 0$) whenever $n \geq 3$. If $n = 2$, we replace the second component of the warped product by $\mathbf{R}$.

Case (a): $\hat{c} = \delta^2 > 0$. In this case, Statement (i) can be proved in exactly the same way as the proof of Statement (i) of Theorem 5.1.

Case (b): $\hat{c} = 0$. In this case, $M^n(0)$ is locally $\mathbf{R} \times_1 E^{n-1}$. Thus,

$$g = dx^2 + du_2^2 + \cdots + du_n^2, \tag{6.3}$$

where $\{u_2, \dots, u_n\}$ is a Euclidean coordinate system on E^{n-1}. We define a symmetric bilinear form σ by

$$\begin{aligned} &\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = 2b\frac{\partial}{\partial x}, \quad \sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u_i}\right) = b\frac{\partial}{\partial u_i}, \\ &\sigma\left(\frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k}\right) = b\delta_{jk}\frac{\partial}{\partial x}, \quad j, k = 2, \dots, n, \end{aligned} \tag{6.4}$$

where $b = \sqrt{-c}$.

By applying (6.4), we may prove that $M^n(0)$ together with the symmetric bilinear form σ satisfies the three conditions given in Theorem A. Therefore, there exists a Lagrangian H-umbilical isometric immersion $\ell\colon M^n(0) \to \mathbf{C}H^n(4c)$ with $h = J\sigma$ as its second fundamental form.

Case (c): $\hat{c} = -\delta^2 < 0$. In this case, $M^n(\hat{c})$ is covered by local coordinate system $\{x, u_2, \dots, u_n\}$ whose metric tensor is given by

$$g = dx^2 + e^{2\delta x}(du_2^2 + \ldots + du_n^2). \tag{6.5}$$

From (6.5) we have

$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}=0,\quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial u_k}=\delta\frac{\partial}{\partial u_k},$$
$$\nabla_{\frac{\partial}{\partial u_j}}\frac{\partial}{\partial u_k}=-\delta\delta_{jk}e^{2\delta x}\frac{\partial}{\partial x},\quad 2\le i,j,k\le n. \tag{6.6}$$

We define a TM-valued symmetric bilinear form σ on $M^n(\hat{c})$ by

$$\sigma\left(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\right)=2b\frac{\partial}{\partial x},\quad \sigma\left(\frac{\partial}{\partial x},\frac{\partial}{\partial u_i}\right)=b\frac{\partial}{\partial u_i},$$
$$\sigma\left(\frac{\partial}{\partial u_j},\frac{\partial}{\partial u_k}\right)=b\delta_{jk}e^{2\delta x}\frac{\partial}{\partial x},\quad j,k=2,\dots,n, \tag{6.7}$$

where $b=\sqrt{\hat{c}-c}$.

By applying (6.6) and (6.7) and by a long computation, we know that $M^n(\hat{c})$ together with the symmetric bilinear form σ satisfies the three conditions given in Existence Theorem (Theorem A). Therefore, there exists a Lagrangian H-umbilical isometric immersion ℓ: $M^n(\hat{c})\to \mathbb{C}H^n(4c)$ with $h=J\sigma$ as its second fundamental form. This proves Statement (i).

Because Statement (ii) can be proved in exactly the same way as that of Statement (ii) of Theorem 5.1, we omit it.

Now, we prove Statement (iii). Assume L: $M\to\mathbb{C}H^n(4c)$ is a Lagrangian H-umbilical isometric immersion whose second fundamental form takes the form of (6.1) for some constant $b\neq 0$. Then M is of constant sectional curvature $\delta^2=b^2+c$. Hence $M^n(\hat{c})$ is locally one of the warped products $I\times_{\frac{1}{\delta}\cos(\delta x)}S^{n-1}(1)$, $\mathbb{R}\times_1 E^{n-1}$, or $I\times_{e^{\delta x}}E^{n-1}$, according to $\hat{c}=\delta^2>0$, $\hat{c}=0$, or $\hat{c}=-\delta^2<0$. When $n=2$, the second component of the warped product shall be replaced by $\mathbb{R}$. In the following, let ϕ: $M\to H_1^{2n+1}(c)\subset\mathbb{C}_1^{n+1}$ denote a horizontal lift of the Lagrangian immersion L: $M\to\mathbb{C}H^n(4c)$.

CASE (iii-1): $\hat{c}=\delta^2>0$. In this case, by applying a method similar to the proof of Statement (iii) of Theorem 5.1, we obtain (6.2-1).

CASE (iii-2): $\hat{c}=0$. In this case, (6.1), (6.3) and Gauss' formula yield

$$\phi_{xx}=2bi\,\phi_x-c\phi,\quad \frac{\partial^2\phi}{\partial u_j\partial x}=ib\frac{\partial\phi}{\partial u_j}, \tag{6.8}$$

$$\frac{\partial^2\phi}{\partial u_j\partial u_k}=\delta_{jk}(ib\phi_x-c\phi),\quad j,k=2,\dots,n \tag{6.9}$$

where $b = \sqrt{-c}$.

By solving the first equation of (6.8), we have

$$\phi(x, u_2, \dots, u_n) = e^{ibx}\{A(u_2, \dots, u_n) + xB(u_2, \dots, u_n)\}, \tag{6.10}$$

for some $\mathbb{C}_1^{n+1}$-valued functions A and B. Applying (6.10), the second equation of (6.8), and $b^2 + c = 0$, we conclude that B is a constant vector. Therefore, by applying (6.9) and (6.10), we obtain

$$\phi = e^{ibx}\left(\frac{i}{2}bB\sum_{j=2}^{n} u_j^2 + \sum_{j=2}^{n}\gamma_j u_j + Bx + C\right), \tag{6.11}$$

where B, γ_j, C are constant vectors. By choosing the following initial conditions:

$$\begin{gathered}\phi(0, \dots, 0) = (1/b, 0, \dots, 0), \quad \phi_x(0, \dots, 0) = (0, 1, 0, \dots, 0),\\ \frac{\partial\phi}{\partial u_2}(0, \dots, 0) = (0, 0, 1, \dots, 0), \dots, \frac{\partial\phi}{\partial u_n}(0, \dots, 0) = (0, \dots, 0, 1),\end{gathered} \tag{6.12}$$

we obtain (6.2-2).

CASE (iii-3): $\hat{c} = b^2 + c < 0$. In this case, we put $\delta = \sqrt{-b^2 - c}$. Form (6.5)–(6.7) and Gauss' formula; we have

$$\phi_{xx} = 2bi\,\phi_x - c\phi, \quad \frac{\partial^2\phi}{\partial u_j \partial x} = (ib + \delta)\frac{\partial\phi}{\partial u_j}, \tag{6.13}$$

$$\frac{\partial^2\phi}{\partial u_j \partial u_k} = \delta_{jk}e^{2\delta x}\{(ib - \delta)\phi_x - c\phi\}, \quad j, k = 2, \dots, n. \tag{6.14}$$

By solving the first equation of (6.13), we get

$$\phi(x, u_2, \dots, u_n) = e^{ibx}\{A(u_2, \dots, u_n)e^{\delta x} + xB(u_2, \dots, u_n)e^{-\delta x}\}, \tag{6.15}$$

for some $\mathbb{C}_1^{n+1}$-valued functions A and B. Applying (6.15), the second equation of (6.13), and the equation $\delta^2 + b^2 + c = 0$, we conclude that B is a constant vector. Therefore, by applying (6.14), (6.15) and a long direct computation, we obtain

$$\phi = e^{ibx}\left(e^{\delta x}(\delta(\delta - bi)B\sum_{j=2}^{n} u_j^2 + \sum_{j=2}^{n}\gamma_j u_j + C) + Be^{-\delta x}\right), \tag{6.16}$$

where B, γ_j and C are constant vectors. By choosing the conditions given by (6.12) as we did for case (iii-2), we obtain (6.2-3). ∎

7. Classification of Lagrangian H-umbilical submanifolds in CP^n

Theorems 4.1, 4.2 and 5.1 imply that there exist abundant examples of Lagrangian H-umbilical submanifolds in $CP^n(4c)$ such that

$$\begin{aligned} &h(e_1,e_1)=\lambda Je_1, \quad h(e_2,e_2)=\cdots=h(e_n,e_n)=\mu Je_1,\\ &h(e_1,e_j)=\mu Je_j, \quad h(e_j,e_k)=0, \quad j\neq k, \quad j,k=2,\ldots,n, \end{aligned} \tag{7.1}$$

for some functions λ,μ with respect to some suitable orthonormal local frame field.

The purpose of this and the next sections is to study and to classify Lagrangian H-umbilical submanifolds in $CP^n(4c)$. Theorem 7.1 says that "most" Lagrangian H-umbilical submanifolds of $CP^n(4c)$ are Lagrangian submanifolds obtained from Legendre curves via warped products as constructed in Theorem 4.1.

We consider the two cases $n\geq 3$ and $n=2$ separately.

THEOREM 7.1: *Let $n\geq 3$ and $\tilde{\psi}$: $M\to CP^n(4c)$ be a Lagrangian H-umbilical isometric immersion.*

(i) *If M is of constant sectional curvature $\hat{c}$, then either*
  (i-1) *$\hat{c}=c$ (cf. Theorem 5.1 for such examples) or*
  (i-2) *$\hat{c}=b^2+c>c$ and up to rigid motions of $CP^n(4c)$, M is isometrically immersed in $CP^n(4c)$ given by (5.2) in Theorem 5.1.*

(ii) *If M contains no open subsets of constant sectional curvature $\geq c$, then there exists a unit speed Legendre curve*

$$z(x)=(z_1(x),z_2(x))\colon I\to S^3(c)\subset \mathbb{C}^2 \tag{7.2}$$

such that up to rigid motions of $CP^n(4c)$, $\tilde{\psi}$ is $\pi\circ\psi$ where ψ is defined by

$$\psi(x,y_1,\ldots,y_n)=(z_1(x),z_2(x)y_1,\ldots,z_2(x)y_n) \tag{7.3}$$

with $y_1^2+\cdots+y_n^2=1$ and π is the projection of Hopf's fibration (cf. Theorem 4.1 for details).

Proof: Let M be a Lagrangian H-umbilical submanifold of $CP^n(4c)$ whose second fundamental form takes the form (7.1) for some functions λ,μ.

CASE (i): M is of constant sectional curvature $\hat{c}$. In this case, because $n \geq 3$, (7.1) implies $\mu(\lambda - 2\mu) = 0$.

CASE (i-1): If μ vanishes identically, then M is of constant curvature c.

CASE (i-2): If $\mu \neq 0$, then $\lambda = 2\mu \neq 0$ on a nonempty open subset V of M. By applying Theorem 5.1 we know that λ and μ are nonzero constants on V. Hence, by continuity, we obtain $V = M$. Put $b = \mu$ and $\hat{c} = \delta^2 = b^2 + c > c$. Then by applying Theorem 5.1 again, we know that up to rigid motions of $\mathbb{C}P^n(4c)$, M is isometrically immersed in $\mathbb{C}P^n(4c)$ by (5.2). This proves Statement (i).

Now, we prove Statement (ii). So, we assume M contains no open subsets of constant sectional curvature $\hat{c}$ ($\geq c$). In this case,

$$U := \{p \in M \colon \mu(\lambda - 2\mu) \neq 0 \text{ at } p\} \tag{7.4}$$

is an open dense subset of M.

Let $e_1, \ldots, e_n$ be an orthonormal local frame field on M satisfying (7.1). Let $\omega^1, \ldots, \omega^n$ denote the dual 1-forms of $e_1, \ldots, e_n$. Denote by (ω_B^A) the connection forms on M defined by

$$\tilde{\nabla} e_i = \sum_{j=1}^n \omega_i^j e_j + \sum_{j=1}^n \omega_i^{j^*} e_{j^*}, \quad \tilde{\nabla} e_{i^*} = \sum_{j=1}^n \omega_{i^*}^j e_j + \sum_{j=1}^n \omega_{i^*}^{j^*} e_{j^*}, \tag{7.5}$$

where $\omega_i^j = -\omega_j^i$, $\omega_{i^*}^{j^*} = -\omega_{j^*}^{i^*}$, $i = 1, \ldots, n$, $A, B = 1, \ldots, n, 1^*, \ldots, n^*$.

For a Lagrangian submanifold M in $\mathbb{C}P^n(4c)$, (2.3) and (2.4) yield

$$\omega_j^{i^*} = \omega_i^{j^*}, \quad \omega_i^j = \omega_{i^*}^{j^*}, \quad \omega_j^{i^*} = \sum_{k=1}^n h_{jk}^i \omega^k. \tag{7.6}$$

From (7.1) and (7.6) we find

$$\omega_1^{1^*} = \lambda\omega^1, \quad \omega_i^{1^*} = \mu\omega^i, \quad \omega_i^{i^*} = \mu\omega^1, \quad \omega_j^{i^*} = 0, \quad 2 \leq i \neq j \leq n. \tag{7.7}$$

By (7.1), (7.7) and equation (2.6) of Codazzi with $X = Z = e_1, Y = e_j$, we obtain

$$e_1\mu = (\lambda - 2\mu)\omega_1^2(e_2) = \cdots = (\lambda - 2\mu)\omega_1^n(e_n), \tag{7.8}$$

$$e_j\lambda = (2\mu - \lambda)\omega_j^1(e_1), \quad j > 1, \tag{7.9}$$

$$(\lambda - 2\mu)\omega_1^j(e_k) = 0, \quad 1 < j \neq k \leq n. \tag{7.10}$$

Similarly, by (7.1), (7.7) and equation (2.6) of Codazzi with $X = Z = e_j, Y = e_1$, we obtain

$$e_j\mu = 3\mu\omega_1^j(e_1), \tag{7.11}$$

$$\mu\omega_1^j(e_1) = 0, \quad j > 1. \tag{7.12}$$

We remark that (7.10) and (7.12) occur only for the case $n \geq 3$.

Since $n \geq 3$, (7.8), (7.11) and (7.12) imply

$$\omega_1^j = \left(\frac{e_1\mu}{\lambda - 2\mu}\right)\omega^j, \quad e_j\lambda = e_j\mu = 0, \quad j = 2, \ldots, n, \tag{7.13}$$

$$\omega_1^j(e_k) = 0, \quad 1 < j \neq k \leq n. \tag{7.14}$$

Let $\mathcal{D}$ denote the distribution spanned by e_1 and $\mathcal{D}^\perp$ be the orthogonal complementary distribution of $\mathcal{D}$, i.e., $\mathcal{D}^\perp$ is spanned by $\{e_2, \ldots, e_n\}$.

LEMMA 7.2: *On U we have*

1. *the integral curves of e_1 are geodesics of M,*
2. *the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are both integrable,*
3. *there exist local coordinate systems $\{x_1, \ldots, x_n\}$ such that* (a) *$\mathcal{D}$ is spanned by $\{\partial/\partial x\}$ and $\mathcal{D}^\perp$ is spanned by $\{\partial/\partial x_2, \ldots, \partial/\partial x_n\}$ and* (b) *$e_1 = \partial/\partial x$, $\omega^1 = dx$, and*
4. *λ and μ are functions of $x = x_1$ satisfying*

$$k'(x) + k^2(x) = \mu^2 - \lambda\mu - c, \qquad k = \frac{\mu'}{\lambda - 2\mu}, \tag{7.15}$$

where $'$ denotes differentiation with respect to x.

Proof: (7.13) and Cartan's structure equations imply $d\omega^1 = 0$ and $\nabla_{e_1} e_1 = 0$. Therefore, the integral curves of e_1 are geodesics. This proves Statement (1).

For $j, k > 1$, (7.14) implies $\langle [e_j, e_k], e_1\rangle = \omega_k^1(e_j) - \omega_j^1(e_k) = 0$ which shows that the distribution $\mathcal{D}^\perp$ is integrable. The integrability of $\mathcal{D}$ is obvious, since $\mathcal{D}$ is a 1-dimensional distribution. This proves Statement (2).

Since $\mathcal{D}$ is 1-dimensional, there exists a local coordinate system $\{y_1, \ldots, y_n\}$ such that $e_1 = \partial/\partial y_1$. Because $\mathcal{D}^\perp$ is integrable, there also exists a local coordinate system $\{z_1, \ldots, z_n\}$ such that $\partial/\partial z_2, \ldots, \partial/\partial z_n$ span $\mathcal{D}^\perp$. Put

$$x_1 = y_1, x_2 = z_2, \ldots, x_n = z_n. \tag{7.16}$$

Then $\{x_1, \ldots, x_n\}$ is a local coordinate system satisfying conditions given in (3).

From (7.12) and statement (3), we see that functions λ and μ depend only on $x\,(=x_1)$. (7.15) follows from (7.13) and the structure equations. This proves Lemma 7.2. ∎

An integrable distribution $\mathcal{F}$ in M is called **spherical** if leaves of $\mathcal{F}$ are totally umbilical submanifolds with parallel mean curvature vector in M.

LEMMA 7.3: *On U the distribution $\mathcal{D}$ is auto-parallel and its orthogonal complementary distribution $\mathcal{D}^{\perp}$ is spherical. Moreover, each leaf of $\mathcal{D}^{\perp}$ is of constant sectional curvature $c+\mu^2+k^2$, where $k=\frac{\mu'}{\lambda-2\mu}$.*

Proof: Lemma 7.2 implies that the distribution $\mathcal{D}$ is auto-parallel. Let X, Y be two vector fields in $\mathcal{D}^{\perp}$ and e_1 a unit vector field in $\mathcal{D}$. Then, by (7.1), (1.3), and Codazzi's equation, we have

$$\begin{aligned}\lambda\langle\nabla_X Y, e_1\rangle &= \langle\nabla_X Y, A_{Je_1}e_1\rangle = -\langle Y, \nabla_X(A_{Je_1}e_1)\rangle\\ &= -\langle Y, (\nabla_X A_{Je_1})e_1\rangle - \langle Y, A_{Je_1}(\nabla_X e_1)\rangle\\ &= -\langle Y, \nabla_{e_1}(A_{Je_1}X)\rangle + \langle Y, A_{Je_1}(\nabla_{e_1}X)\rangle\\ &\quad -\langle Y, A_{Je_1}(\nabla_X e_1)\rangle - \langle Y, A_{D_X Je_1}e_1\rangle\\ &= -\langle Y, \nabla_{e_1}(\mu X)\rangle + 2\mu\langle Y, \nabla_{e_1}X\rangle - \mu\langle Y, \nabla_X e_1\rangle\\ &= -(e_1\mu)\langle X, Y\rangle + 2\mu\langle\nabla_X Y, e_1\rangle.\end{aligned}$$

which yields

$$\langle\nabla_X Y, e_1\rangle = \left(\frac{e_1\mu}{2\mu-\lambda}\right)\langle X, Y\rangle. \tag{7.17}$$

Formula (7.17) implies that leaves of $\mathcal{D}^{\perp}$ are totally umbilical hypersurfaces with parallel mean curvature vector. Therefore, $\mathcal{D}^{\perp}$ is a spherical distribution.

From Lemma 7.2, (7.17) and Gauss' equation, we know that each leaf of $\mathcal{D}^{\perp}$ is of constant sectional curvature $c+\mu^2+k^2$. This proves Lemma 7.3. ∎

LEMMA 7.4: *U is an open portion of the warped product $I \times_{\frac{1}{\ell(x)}} S^{n-1}(1)$ where*

$$\ell = \sqrt{c+\mu^2+k^2} \quad \text{and} \quad k = \frac{\mu'}{\lambda-2\mu}. \tag{7.18}$$

Proof: By applying Lemma 7.3 and a result of Hiepko [9], we know that U is locally a warped product $I \times_{f(x)} S^{n-1}(1)$, where $S^{n-1}(1)$ is the unit $(n-1)$-sphere and $f(x)$ is the warped function. Moreover, we also know that each vector tangent to I is in the distribution $\mathcal{D}$ and each vector tangent to $S^{n-1}(1)$ is in the complementary distribution $\mathcal{D}^{\perp}$.

With respect to a spherical coordinate system $\{u_2, \dots, u_n\}$ on $S^{n-1}(1)$, we have

$$g_0 = du_2^2 + \cos^2 u_2 du_3^2 + \cdots + \cos^2 u_2 \cdots \cos^2 u_{n-1} du_n^2.$$

Thus

$$g = dx^2 + f^2(x)\{du_2^2 + \cos^2 u_2 du_3^2 + \cdots + \cos^2 u_2 \cdots \cos^2 u_{n-1} du_n^2\}. \tag{7.19}$$

From (7.19) we obtain

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial u_k} = \frac{f'}{f} \frac{\partial}{\partial u_k}, \quad \nabla_{\frac{\partial}{\partial u_2}} \frac{\partial}{\partial u_2} = -ff' \frac{\partial}{\partial x}, \\ &\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = -\tan u_i \frac{\partial}{\partial u_j}, \quad 2 \le i < j, \\ &\nabla_{\frac{\partial}{\partial u_j}} \frac{\partial}{\partial u_j} = -ff' \prod_{\ell=2}^{j-1} \cos^2 u_\ell \frac{\partial}{\partial x} + \sum_{k=2}^{j-1} \left(\frac{\sin 2u_k}{2} \prod_{\ell=k+1}^{j-1} \cos^2 u_\ell \right) \frac{\partial}{\partial u_k}, \end{aligned} \tag{7.20}$$

where $2 \le i, j, k \le n$.

(7.1), (7.20) and Codazzi's equation imply

$$\frac{f'}{f} = k, \qquad k = \frac{\mu'}{\lambda - 2\mu}. \tag{7.21}$$

Thus, there is a nonzero constant b such that

$$f = be^{\int k(x)dx}. \tag{7.22}$$

By applying (7.20), we know that the sectional curvature of the plane section spanned by $\partial/\partial u_2, \partial/\partial u_3$ is given by

$$K\left(\frac{\partial}{\partial u_2} \wedge \frac{\partial}{\partial u_3}\right) = \frac{1}{b^2} e^{-2\int k(x)dx} - k^2. \tag{7.23}$$

On the other hand, (7.1) and the equation of Gauss yield

$$K\left(\frac{\partial}{\partial u_2} \wedge \frac{\partial}{\partial u_3}\right) = c + \mu^2. \tag{7.24}$$

Therefore we have

$$f(x) = be^{\int k(x)dx} = \frac{1}{\ell}, \qquad \ell(x) = \sqrt{c + \mu^2 + k^2}. \tag{7.25}$$ ∎

We need the following.

LEMMA 7.5: *Let $\lambda(x)$ and $\mu(x)$ be real-valued functions of x and $k(x) = \frac{\mu'}{\lambda - 2\mu}$. If λ and μ satisfy*

$$k' + k^2 - \mu^2 + \lambda\mu + c = 0, \tag{7.26}$$

then

$$y_1 = e^{\int (k+i\mu)dx}, \quad y_2 = y_1 \int \exp\left(\int^x \{i\lambda(t) - 2i\mu(t) - 2k(t)\}\, dt\right) dx \tag{7.27}$$

are two independent complex-valued solutions of the differential equation

$$y''(x) = i\lambda(x)y'(x) - cy(x). \tag{7.28}$$

Proof: This lemma can be verified by a direct computation. ∎

Now we return to the proof of Statement (ii) of Theorem 7.1.

Consider the restriction of $\bar{\psi}$ to the open dense subset U. We denote this restricted Lagrangian isometric immersion also by $\bar{\psi}$. Let ψ: $\hat{U} \to S^{2n+1}(c) \subset \mathbb{C}^{n+1}$ be a horizontal lift of the Lagrangian immersion $\hat{\psi}$: $U \to \mathbb{C}P^n(4c)$ via Hopf's fibration. Then, by (7.1), (7.19), (7.20), (7.21) and Gauss' formula, we have

$$\psi_{xx} = \lambda i\, \psi_x - c\psi, \quad \psi_x = \frac{\partial \psi}{\partial x}, \quad \psi_{xx} = \frac{\partial^2 \psi}{\partial x^2}, \tag{7.29}$$

$$\tilde{\nabla}_Y \psi_x = (i\mu + k)Y, \tag{7.30}$$

$$\tilde{\nabla}_Y \tilde{\nabla}_Z \psi = \langle Y, Z\rangle \{(i\mu)\psi_x - c\psi\} + \nabla_Y Z, \tag{7.31}$$

where Y, Z are vector fields tangent to the second component $S^{n-1}(1)$ of the warped product and $\nabla_Y Z$ is the tangential component of $\tilde{\nabla}_Y Z$.

By Lemma 7.2, Lemma 7.5 and (7.29) we have

$$\begin{aligned}\psi(x, u_2, \ldots, u_n) &= A(u_2, \ldots, u_n)e^{\int (k+i\mu)dx} \\ &\quad + B(u_2, \ldots, u_n)e^{\int (k+i\mu)dx} \int \left(e^{\int^x \{i\lambda(t) - 2i\mu(t) - 2k(t)\}dt}\right) dx,\end{aligned} \tag{7.32}$$

for some $\mathbb{C}^{n+1}$-valued vector functions A and B. From (7.30) and (7.32), we obtain $\partial B/\partial u_j = 0$, $j = 2, \ldots, n$. Thus, B is a constant vector in $\mathbb{C}^{n+1}$.

By applying (7.32), (7.31) with $Y = Z = \partial/\partial u_2$, we may obtain

$$\begin{aligned}\psi &= (c_1(u_3, \ldots, u_n)\sin u_2 + A_2(u_3, \ldots, u_n)\cos u_2)e^{\int (i\mu+k)dx} \\ &\quad + b^2(i\mu - k)Be^{\int (i\lambda - i\mu + k)dx},\end{aligned} \tag{7.33}$$

where c_1, A_2 and B are orthogonal with $|c_1| = |A_2|$. Then, by applying (7.33) and (7.31) with $Y = \partial/\partial u_2$, $Z = \partial/\partial u_j$, $j = 3, \dots, n$, we conclude that c_1 is a constant vector.

Furthermore, by applying (7.33) and (7.31) with $Y = Z = \partial/\partial u_3$, we have

$$A_2 = (\sin u_3)c_2(u_4, \dots, u_n) + (\cos u_3)A_3(u_4, \dots, u_n). \tag{7.34}$$

By applying (7.33), (7.34) and (7.31) with $Y = \partial/\partial u_3$, $Z = \partial/\partial u_j, j = 4, \dots, n$, we also know that c_2 is a constant vector.

Continue such processes $n-1$ times, we obtain

$$\begin{aligned}\psi =&(c_1 \sin u_2 + c_2 \sin u_3 \cos u_2 + \cdots + c_{n-1} \sin u_{n-1} \prod_{k=2}^{n-2} \cos u_k \\ &+ c_n \prod_{k=2}^{n-1} \cos u_k)e^{\int (i\mu+k)dx} + b^2(i\mu - k)Be^{\int (i\lambda - i\mu+k)dx},\end{aligned} \tag{7.35}$$

where $c_1, \dots, c_n, B$ are orthogonal constant vectors. By choosing $B, c_1, \dots, c_n$ parallel to the canonical basis $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$, respectively, we obtain the desired expression (7.3) for ψ. ∎

8. Lagrangian H-umbilical surfaces in $\mathbb{C}P^2$

In this section we investigate Lagrangian H-umbilical surfaces in $\mathbb{C}P^2$.

THEOREM 8.1: *We have the following results.*

(i) *Let $\bar{\psi}: M \to \mathbb{C}P^2(4c)$ be a Lagrangian isometric immersion from a surface into $\mathbb{C}P^2(4c)$. Then, for each point $p \in M$, there exists an orthonormal basis $\{e_1, e_2\}$ of T_pM such that the second fundamental form of $\bar{\psi}$ at p takes the form:*

$$h(e_1, e_1) = \lambda J e_1, \quad h(e_1, e_2) = \mu J e_2, \quad h(e_2, e_2) = \mu J e_1 + \eta J e_2. \tag{8.1}$$

In particular, if M is a Lagrangian minimal surface in $\mathbb{C}P^2(4c)$, for each $p \in M$, there exists an orthonormal basis $\{e_1, e_2\}$ of T_pM such that

$$h(e_1, e_1) = -\mu J e_1, \quad h(e_1, e_2) = \mu J e_2, \quad h(e_2, e_2) = \mu J e_1. \tag{8.2}$$

(ii) *For each constant $\hat{c}$, there exists a non-totally geodesic Lagrangian H-umbilical isometric immersion*

$$\ell: M(\hat{c}) \to \mathbb{C}P^2(4c) \tag{8.3}$$

of a surface of constant curvature $\hat{c}$ into $\mathbb{C}P^2(4c)$ such that integral curves of JH are geodesics in $M(\hat{c})$.

(iii) *Let $\bar{\psi}$: $M \to \mathbb{C}P^2(4c)$ be a Lagrangian H-umbilical isometric immersion of a surface into $\mathbb{C}P^2(4c)$. If M contains no open subsets of constant sectional curvature $\geq c$ and if integral curves of JH are geodesics in M, then there exists a unit speed Legendre curve*

$$z(x) = (z_1(x), z_2(x)) \colon I \to S^3(c) \subset \mathbb{C}^2 \tag{8.4}$$

such that up to rigid motions of $\mathbb{C}P^2(4c)$, $\bar{\psi}$: $M \to \mathbb{C}P^2(4c)$ is $\pi \circ \psi$ where ψ defined by

$$\psi(x,u) = (z_1(x), z_2(x)\sin u, z_2(x)\cos u). \tag{8.5}$$

Proof: (i) If p is a totally geodesic point, there is nothing to prove. So we assume that p is a non-totally geodesic point. We define a function β_p by

$$\beta_p \colon UM_p \to \mathbb{R} \colon v \mapsto \beta_p(v) = \langle h(v,v), Jv \rangle,$$

where $UM_p = \{v \in T_pM \colon \langle v,v \rangle = 1\}$. Since UM_p is a compact set, there exists a vector v in UM_p such that β_p attains an absolute minimum at v. Since p is non-totally geodesic, it follows from (2.4) that $\beta_p \neq 0$. By linearity, we have $\beta_p(v) < 0$. Because β_p attains an absolute minimum at v, it follows from (2.4) that $\langle h(v,v), Jw \rangle = 0$, for all w orthogonal to v. So, using (2.4), v is an eigenvector of the symmetric operator A_{Jv}. By choosing an orthonormal basis $\{e_1, e_2\}$ of T_pM with $e_1 = v$, it is easy to see that $\{e_1, e_2\}$ satisfies the desired property. This proves Statement (i).

(ii) We divide the proof of Statement (ii) into five subcases.

Case (ii-1): $\hat{c} > c$. Put $\hat{c} = \delta^2$, $b = \sqrt{\hat{c} - c}$. We define a C-totally real isometric immersion $\phi_{\hat{c}} \colon I \times_{\frac{1}{\delta}\cos(\delta x)} \mathbb{R} \to S^5(c) \subset \mathbb{C}^3$ by

$$\begin{aligned} \phi_{\hat{c}}(x,y) = \frac{e^{ibx}}{2\delta^2}\Big(&(\frac{b(b-\delta)}{\sqrt{c}} + \sqrt{c}\cos y)e^{i\delta x} + (\frac{b(b+\delta)}{\sqrt{c}} + \sqrt{c}\cos y)e^{-i\delta x}, \\ &(\delta - b + b\cos y)e^{i\delta x} - (\delta + b - b\cos y)e^{-i\delta x}, \delta(e^{i\delta x} + e^{-i\delta x})\sin y\Big). \end{aligned} \tag{8.6}$$

By a direct and long computation we know that $\pi \circ \phi_{\hat{c}}$ defines a Lagrangian H-umbilical isometric immersion from the surface $M(\hat{c})$ into $\mathbb{C}P^2(4c)$. It is easy to verify that integral curves of JH are geodesics in M.

CASE (ii-2): $\hat{c} = c$. Consider the warped product $M(c) = I \times_{\frac{1}{\sqrt{c}}\cos(\sqrt{c}x)} \mathbb{R}$, where I is the open interval $(-\frac{\pi}{2\sqrt{c}}, \frac{\pi}{2\sqrt{c}})$. We have

$$(8.7)\quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = 0, \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = -\sqrt{c}\tan(\sqrt{c}x)\frac{\partial}{\partial y}, \quad \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = \frac{\sin(2\sqrt{c}x)}{2\sqrt{c}}\frac{\partial}{\partial x}.$$

We define a symmetric bilinear form σ on $M(c)$ by

$$(8.8)\quad \begin{aligned} &\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \sec(\sqrt{c}x)\frac{\partial}{\partial x}, \quad \sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \sec(\sqrt{c}x)\frac{\partial}{\partial y}, \\ &\sigma\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \sec(\sqrt{c}x)\frac{\partial}{\partial x}. \end{aligned}$$

By applying (8.7), (8.8) and by a long computation, we know that $M(c)$ together with the symmetric bilinear form σ satisfies the three conditions given in Theorem A. Therefore, there exists a Lagrangian H-umbilical isometric immersion ℓ: $M(c) \to \mathbb{C}P^n(4c)$ with $h = J\sigma$ as its second fundamental form. In this case integral curves of JH are also geodesics in M.

CASE (ii-3): $0 < \hat{c} \le c$. Put $\hat{c} = \delta^2$. Consider the warped product surface $M = I \times_{\frac{1}{\delta}\cos(\delta x)} \mathbb{R}$, where I is the open interval $(-\frac{\pi}{2\delta}, \frac{\pi}{2\delta})$. Let

$$(8.9)\quad e_1 = \frac{\partial}{\partial x}, \quad e_2 = \delta\sec(\delta x)\frac{\partial}{\partial y}.$$

Then e_1, e_2 form an orthonormal frame field and

$$(8.10)\quad \begin{aligned} &\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = 0, \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = -\delta\tan(\delta x)\frac{\partial}{\partial y}, \quad \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = \frac{\sin(2\delta x)}{2\delta}\frac{\partial}{\partial x}, \\ &\omega_2^1 = \delta\tan(\delta x)\omega^2. \end{aligned}$$

Let a be a real number greater than $2(c-\delta^2)$. Then there is an open subinterval $\hat{I}$ of I such that $a\sec^2(\delta x) > 2(c-\delta^2)$. On $\hat{I}$ we define functions λ and μ by

$$(8.11)\quad \lambda = \frac{a\sec^2(\delta x) - 4c + 4\delta^2}{\sqrt{2a\sec^2(\delta x) - 4c + 4\delta^2}}, \quad \mu = \frac{1}{2}\sqrt{2a\sec^2(\delta x) - 4c + 4\delta^2}.$$

Then we have

$$(8.12)\quad \lambda'(x) = \frac{2\delta(\lambda^2 + 4c - 4\delta^2)\tan(\delta x)}{\sqrt{\lambda^2 + 4c - 4\delta^2} + \lambda}.$$

We define a symmetric bilinear form σ on $M(\hat{c})$ by

$$\sigma(e_1,e_1)=\lambda e_1,\quad \sigma(e_1,e_2)=\mu e_2,\quad \sigma(e_2,e_2)=\mu e_1. \tag{8.13}$$

It follows from (8.9)–(8.13) and a long computation that $M(\hat{c})=\hat{I}\times_{\frac{1}{\delta}\cos(\delta x)}\mathbf{R}$ together with σ satisfies the three conditions given in Theorem A. Hence, there exists a Lagrangian H-umbilical isometric immersion ℓ: $M(\hat{c})\to CP^n(4c)$ with $h=J\sigma$ as its second fundamental form. It is easy to see that integral curves of JH are geodesics in M.

CASE (ii-4): $\hat{c}=0$. Let b be a nonnegative number. We define a map ψ_b: $\mathbf{R}\times\mathbf{R}\to S^5(c)\subset\mathbf{C}^3$ by

$$\begin{aligned}\psi_b(x,y)=\frac{e^{ibx}}{2\alpha}\Bigg(&\frac{e^{i\alpha x}}{2\sqrt{c}}((\alpha-b)(e^{\gamma iy}+e^{-\gamma iy})+\frac{e^{-i\alpha x}}{\sqrt{c}}(b+\alpha)),\\ &\frac{e^{i\alpha x}}{2}(e^{\gamma iy}+e^{-\gamma iy})-\frac{e^{-i\alpha x}}{\sqrt{c}},\frac{\sqrt{\alpha}e^{i\alpha x}}{\sqrt{2(\alpha+b)}}(e^{\gamma iy}-e^{-\gamma iy})\Bigg),\end{aligned} \tag{8.14}$$

where $\alpha=\sqrt{b^2+c}$ and $\gamma=\sqrt{2\alpha(\alpha+b)}$.

By a direct but long computation we know that, for each $b\geq 0$, the map $\pi\circ\psi_b$ defines a Lagrangian H-umbilical isometric immersion from a Euclidean 2-plane $\mathbf{E}^2$ into $CP^2(4c)$ such that the integral curves of JH are geodesics.

CASE (ii-5): $\hat{c}<0$. Put $\hat{c}=-\delta^2$ with $\delta>0$. Consider the warped product $M=\mathbf{R}\times_{\frac{1}{\delta}e^{\delta x}}\mathbf{R}$. Then M is of constant curvature $\hat{c}<0$. Let

$$e_1=\frac{\partial}{\partial x},\quad e_2=\delta e^{-\delta x}\frac{\partial}{\partial y}. \tag{8.15}$$

Then e_1,e_2 form an orthonormal frame field and, moreover, we have $\omega_2^1=-\delta\omega^2$.

Let a be a real number less than 2. Then there is an open interval I containing 0 such that $2>ae^{2\delta x}$. On I we define functions λ and μ by

$$\lambda=\frac{2\sqrt{c+\delta^2}(ae^{2\delta x}-1)}{e^{\delta x}\sqrt{2a-a^2e^{2\delta x}}},\quad \mu=-\frac{\sqrt{c+\delta^2}\sqrt{2a-a^2e^{2\delta x}}}{ae^{\delta x}}. \tag{8.16}$$

Then we have

$$\lambda'(x)=\frac{2\delta(\lambda^2+4c+4\delta^2)}{\sqrt{\lambda^2+4c+4\delta^2}-\lambda}. \tag{8.17}$$

We define a symmetric bilinear form σ on $M(\hat{c})$ by

$$\sigma(e_1,e_1)=\lambda e_1,\quad \sigma(e_1,e_2)=\mu e_2,\quad \sigma(e_2,e_2)=\mu e_1. \tag{8.18}$$

By (8.15)–(8.18) and a long computation, we know that $M(\hat{c}) = I \times_{\frac{1}{\delta}e^{\delta x}} \mathbb{R}$ together with σ satisfies the three conditions given in Theorem A. Thus, there exists a Lagrangian H-umbilical isometric immersion ℓ: $M(\hat{c}) \to \mathbb{C}P^n(4c)$ with $h = J\sigma$ as its second fundamental form. It is easy to verify that integral curves of JH are geodesics in M. This proves Statement (ii).

Statement (iii) can be proved in a way similar to that of Statement (ii) of Theorem 7.1 with minor modifications. ∎

Remark 8.2: The assumption on the integral curves of JH made in Statement (iii) of Theorem 8.1 cannot be omitted. This can be seen from Statement (i) of Theorem 8.1 and the fact that the only Lagrangian minimal surfaces of constant curvature > 0 in $\mathbb{C}P^2(4c)$ is the totally geodesic one (cf. [7]) and that most Lagrangian minimal surfaces in $\mathbb{C}P^2(4c)$ cannot be expressed in the form of (8.5).

Remark 8.3: From the proof of case (ii-2) in Statement (ii) we know that there exists many Lagrangian isometric immersions from a surface $M(c)$ of constant curvature c into $\mathbb{C}P^2(4c)$ which are different from the totally geodesic one.

9. Lagrangian H-umbilical submanifolds in $\mathbb{C}H^n$

In this and the next sections we investigate Lagrangian H-umbilical submanifolds of $\mathbb{C}H^n$. We assume that the second fundamental form of M takes the following form:

$$\begin{aligned} &h(e_1,e_1)=\lambda Je_1,\quad h(e_2,e_2)=\cdots=h(e_n,e_n)=\mu Je_1,\\ &h(e_1,e_j)=\mu Je_j,\quad h(e_j,e_k)=0,\quad j\neq k,\quad j,k=2,\dots,n, \end{aligned} \tag{9.1}$$

for some functions λ, μ with respect to some orthonormal local frame field.

THEOREM 9.1: *Let $\bar{\psi}$: $M \to \mathbb{C}H^n(4c), c < 0$ be a Lagrangian H-umbilical isometric immersion. If $n \geq 3$, then the following statements hold.*

(i) *If M is of constant sectional curvature $\hat{c}$, then either*

(i-1) *$\hat{c} = c$ (cf. Theorem 6.1, in particular (6.2-3) with $b = 0$) or*

(i-2) *$\hat{c} > c$ and up to rigid motions of $\mathbb{C}H^n(4c)$, M is isometrically immersed in $\mathbb{C}H^n(4c)$ by (6.2-1), (6.2-2) or (6.2-3) given in Theorem 6.1 with $b > 0$.*

(ii) *If M contains no open subset of constant sectional curvature $\geq c$, then M is foliated by real-space-forms $N^{n-1}(u(x))$ of constant sectional curvature $u(x) = c + \mu^2(x) + k^2(x)$ where $k = \frac{\mu'}{\lambda - 2\mu}$ and λ, μ are given by (9.1). Moreover,*

(ii-1) *if $u > 0$, then there exists a unit speed Legendre curve*

$$z = (z_1, z_2)\colon I \to H_1^3(c) \subset \mathbb{C}_1^2$$

such that up to rigid motions of $\mathbb{C}H^n(4c)$, $\bar{\psi}$ is locally given by

$$\pi \circ \psi\colon M \to H_1^{2n+1}(c) \to \mathbb{C}H^n(4c)$$

where

$$\psi(x, y_1, \dots, y_n) = (z_1(x), z_2(x)y_1, \dots, z_2(x)y_n) \tag{9.2}$$

with $y_1^2 + y_2^2 + \cdots + y_n^2 = 1$;

(ii-2) *if $u < 0$, then there exists a unit speed Legendre curve*

$$z = (z_1, z_2)\colon I \to H_1^3(c) \subset \mathbb{C}_1^2$$

such that up to rigid motions of $\mathbb{C}H^n(4c)$, $\bar{\psi}$ is locally given by $\pi \circ \psi$ where

$$\psi(x, y_1, \dots, y_n) = (z_1(x)y_1, \dots, z_1(x)y_n, z_2(x)) \tag{9.3}$$

with $y_1^2 - y_2^2 - \cdots - y_n^2 = 1$ (cf. Theorem 4.2);

(ii-3) *if $u = 0$, then up to rigid motions of $\mathbb{C}H^n(4c)$, $\bar{\psi}$ is locally given by $\pi \circ \psi$ where*

(9.4)

$$\psi = e^{\int_0^x (i\mu + k)dx} \left(\frac{1}{\sqrt{-c}} \left(1 - \frac{c}{2} \sum_{j=2}^n u_j^2 - \int_0^x (i\mu + k) e^{-\int_0^x 2k(t)dt} dx \right), \right.$$
$$\left. (i\mu(0) - k(0)) \left(\frac{1}{2} \sum_{j=2}^n u_j^2 + \frac{1}{c} \int_0^x (i\mu + k) e^{-\int_0^x 2k(t)dt} dx \right), u_2, \dots, u_n \right).$$

Proof: Statement (i) can be proved exactly in the same way as Statement (i) of Theorem 8.1.

For Statement (ii), first we observe that under the same notations as section 7, we have Lemmas 7.2 and 7.3 as well. This implies that M is foliated by real-space-forms $N^{n-1}(u(x))$ of constant sectional curvature $u(x) = c+\mu^2(x)+k^2(x)$ where $k = \frac{\mu'}{\lambda-2\mu}$. We separate the proof of the remaining part of Statement (ii) into three subcases.

CASE (ii-1): $u(x) = c+\mu^2(x)+k^2(x) = \ell^2(x) > 0$. This case can be proved exactly in the same way as Statement (ii) of Theorem 8.1.

CASE (ii-2): $u(x) = c+\mu^2(x)+k^2(x) = -\ell^2(x) < 0$. In this case, Lemma 7.3 and Hiepko's result imply that M is locally a warped product $I \times_{f(x)} H^{n-1}(-1)$. By using Codazzi's equation we may obtain as in the proof of Lemma 7.4 that $f(x) = be^{\int k(x)dx} = 1/\ell(x)$, where b is a positive number.

Let $\psi\colon M \to H_1^{2n+1}(c) \subset \mathbb{C}_1^{n+1}$ be a horizontal lift of the Lagrangian immersion $\bar{\psi}\colon M \to \mathbb{C}H^n(4c)$. Thus, by (9.1) and Gauss' formula, we have

$$\psi_{xx} = \lambda i\,\psi_x - c\psi, \quad \psi_x = \frac{\partial\psi}{\partial x}, \quad \psi_{xx} = \frac{\partial^2\psi}{\partial x^2}, \tag{9.5}$$

$$\tilde{\nabla}_Y\psi_x = (i\mu+k)Y, \tag{9.6}$$

$$\tilde{\nabla}_Y\tilde{\nabla}_Z\psi = \langle Y,Z\rangle\{(i\mu)\psi_x - c\psi\} + \nabla_Y Z, \tag{9.7}$$

where Y, Z are vector fields tangent to the second component $S^{n-1}(1)$ of the warped product and $\nabla_Y Z$ is the tangential component of $\tilde{\nabla}_Y Z$.

We choose coordinates on $I \times_{f(x)} H^{n-1}(-1)$ with

$$g = dx^2 + f^2(x)\{dy^2 + \sinh^2 y(du_3^2 + \cos^2 u_3 du_4^2 + \cdots + \prod_{k=3}^{n-1}\cos^2 u_k du_{n-1}^2)\}, \tag{9.8}$$

for $y > 0$. From (9.8) we have

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = 0, \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = k\frac{\partial}{\partial y}, \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial u_k} = k\frac{\partial}{\partial u_k}, \quad 3 \le k \le n,\\
&\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial u_j} = \coth y\frac{\partial}{\partial u_j}, \quad \nabla_{\frac{\partial}{\partial u_j}}\frac{\partial}{\partial u_k} = -\tan u_j\frac{\partial}{\partial u_k}, \quad 3 \le j < k,\\
&\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = -b^2ke^{2\int kdx}\frac{\partial}{\partial x},\\
&\nabla_{\frac{\partial}{\partial u_3}}\frac{\partial}{\partial u_3} = -b^2ke^{2\int kdx}\sinh^2 y\frac{\partial}{\partial x} - \sinh y\cosh y\frac{\partial}{\partial y},
\end{aligned} \tag{9.9}$$

$$\nabla_{\frac{\partial}{\partial u_j}}\frac{\partial}{\partial u_j} = -\sinh^2 y\left\{b^2ke^{2\int k dx}\prod_{\ell=3}^{j-1}\cos^2 u_\ell\frac{\partial}{\partial x}+\sum_{\ell=3}^{j-1}\cos^2 u_\ell\frac{\partial}{\partial y}\right.$$
$$\left.-\sum_{k=3}^{j-1}\left(\sin u_k\cos u_k\prod_{\ell=k+1}^{j-1}\cos^2 u_\ell\right)\frac{\partial}{\partial u_k}\right\},\quad j\geq 4.$$

By Lemma 7.2, Lemma 7.5 and (9.5) we have

$$\begin{aligned}\psi(x,u_2,\ldots,u_n) &= A(y,u_3,\ldots,u_n)e^{\int_0^x(k+i\mu)dx}\\ &+ B(y,u_3,\ldots,u_n)e^{\int_0^x(k+i\mu)dx}\int_0^x\left(e^{\int_0^x\{i\lambda(t)-2i\mu(t)-2k(t)\}dt}\right)dx,\end{aligned}\tag{9.10}$$

for some $\mathbb{C}^{n+1}$-valued vector functions A and B. From (9.6) and (9.10), we know that B is a constant vector in $\mathbb{C}^{n+1}$.

By applying (9.10) and (9.7) with $Y=Z=\partial/\partial y$, we obtain

$$\begin{aligned}\psi &= (c_1(u_3,\ldots,u_n)\cosh y + A_2(u_3,\ldots,u_n)\sinh y)e^{\int_0^x(i\mu+k)dx}\\ &- b^2(i\mu-k)Be^{\int_0^x(i\lambda-i\mu+k)dx},\end{aligned}\tag{9.11}$$

where c_1, A_2 and B are orthogonal with $|c_1|=|A_2|$. Then, by applying (9.11) and (9.7) with $Y=\partial/\partial y$, $Z=\partial/\partial u_j, j=3,\ldots,n$, we conclude that c_1 is also a constant vector. Furthermore, by applying (9.11) and (9.7) with $Y=Z=\partial/\partial u_3$, we obtain

$$A_2 = c_2(u_4,\ldots,u_n)\sin u_3 + A_3(u_4,\ldots,u_n)\cos u_3.\tag{9.12}$$

Thus, we have

$$\begin{aligned}\psi &= (c_1\cosh y + c_2\sinh y\sin u_3 + A_3\sinh y\cos u_3)e^{\int_0^x(i\mu+k)dx}\\ &- b^2(i\mu-k)Be^{\int_0^x(i\lambda-i\mu+k)dx},\end{aligned}\tag{9.13}$$

where c_1, c_2, B are constant vectors and, moreover, c_1, c_2, A_3, B are orthogonal.

Again, by applying (9.7) with $Y=\partial/\partial u_3$ and $Z=\partial/\partial u_j$, $j>3$, we know that c_2 is a constant vector. Continue such processes $n-1$ times, we obtain

$$\begin{aligned}\psi &= (c_1\cosh u_2 + \sinh u_2(c_2\sin u_3 + c_3\cos u_3\sin u_4 + \cdots\\ &\cdots + c_n\cos u_3\cdots\cos u_{n-1}))e^{\int_0^x(i\mu+k)dx} - b^2(i\mu-k)Be^{\int_0^x(i\lambda-i\mu+k)dx},\end{aligned}\tag{9.14}$$

where $c_1,\ldots,c_n,B$ are constant vectors. Therefore, by choosing a suitable complex coordinate system on $\mathbb{C}_1^{n+1}$ we obtain (9.2) associated with a suitable unit speed Legendre curve $z(x)=(z_1(x),z_2(x))$ in $H_1^3(c)$.

CASE (ii-3): $u(x) = c+\mu^2(x)+k^2(x) = 0$. In this case, Lemma 7.3 and Hiepko's result imply that M is locally a warped product $I \times_{f(x)} E^{n-1}$ with

$$g = dx^2 + f^2(x)\{du_2^2 + du_3^2 + \cdots + du_n^2\}. \tag{9.15}$$

By using Codazzi's equation we may obtain as in the proof of Lemma 7.4 that

$$f(x) = be^{\int_0^x k(x)dx} = \frac{1}{\ell(x)},$$

where b is a positive number. Without loss of generality, we may choose $b = 1$ and so we have

$$f(x) = e^{\int_0^x k(t)dt}.$$

Let ψ: $M \to H_1^{2n+1}(c) \subset \mathbb{C}_1^{n+1}$ be a horizontal lift of the Lagrangian immersion $\bar{\psi}$: $M \to \mathbb{C}H^n(4c)$. Thus, by (9.1) and Gauss' formula, we have

$$\psi_{xx} = \lambda i\,\psi_x - c\psi, \quad \psi_x = \frac{\partial\psi}{\partial x}, \quad \psi_{xx} = \frac{\partial^2\psi}{\partial x^2}, \tag{9.16}$$

$$\psi_{xu_j} = (i\mu + k)\psi_{u_j}, \tag{9.17}$$

$$\psi_{u_j u_k} = \delta_{jk}b^2e^{2\int kdx}\{(i\mu - k)\psi_x - c\psi\}, \quad j,k = 2,\ldots,n. \tag{9.18}$$

By Lemma 7.2, Lemma 7.5 and (9.16) we have

$$\begin{aligned}\psi(x,u_2,\ldots,u_n) &= A(u_2,\ldots,u_n)e^{\int_0^x (k+i\mu)dx} \\ &\quad + B(u_2,\ldots,u_n)e^{\int_0^x (k+i\mu)dx}\int_0^x \left(e^{\int_0^x \{i\lambda(t)-2i\mu(t)-2k(t)\}dt}\right)dx,\end{aligned} \tag{9.19}$$

for some $\mathbb{C}^{n+1}$-valued vector functions A and B. From (9.17) and (9.19), we know that B is a constant vector.

By applying (9.19) and (9.18), we may obtain

$$\frac{\partial A}{\partial u_j \partial u_k} = \delta_{jk}b^2(i\mu - k)Be^{i\int_0^x (\lambda-2\mu)dx}, \quad i,k = 2,\ldots,n. \tag{9.20}$$

On the other hand, $\mu' = k(\lambda - 2\mu)$ and $c + \mu^2 + k^2 = 0$ imply that

$$(i\mu - k)e^{i\int_0^x (\lambda-2\mu)dx}$$

is a constant which is denoted by α. Thus (9.20) yields

$$\frac{\partial A}{\partial u_j \partial u_k} = \delta_{jk}b^2\alpha B, \quad i,k = 2,\ldots,n. \tag{9.21}$$

Hence, A takes the following form:

$$A(u_2,\dots,u_n)=\gamma+\sum_{j=2}^{n}(c_ju_j+\beta_ju^2), \tag{9.22}$$

for some constant vectors γ, c_j, β_j. Combining (9.19) and (9.22) we obtain

$$\psi=e^{\int(i\mu+k)dx}\left(\gamma+\sum_{j=2}^{n}(c_ju_j+\beta_ju^2)\right)+E(x), \tag{9.23}$$

where

$$E(x)=Be^{\int_0^x(i\mu+k)dx}\int_0^x e^{\int_0^x(i\lambda-i\mu+k)dt}dx. \tag{9.24}$$

By applying (9.18), (9.23) and $c+\mu^2+k^2=0$, we obtain

$$E'(x)-(i\mu+k)E(x)=\frac{2\beta_j}{cb^2}(i\mu+k)e^{\int_0^x(i\mu-k)dx}, \tag{9.25}$$

for $j=2,\dots,n$. Hence $\beta_2=\cdots=\beta_n$. We put $\beta=2\beta_j/cb^2$. Then, by solving (9.25), we obtain

$$E(x)=e^{\int(i\mu+k)dx}\left(\beta\int_0^x(i\mu+k)e^{-2\int_0^x k(t)dt}dx+\epsilon\right), \tag{9.26}$$

where ϵ is a constant vector. By combining (9.24) and (9.27) we have

$$\psi=e^{\int_0^x(i\mu+k)dx}\left\{\eta+\sum_{j=2}^{n}c_ju_j+\frac{cb^2\beta}{2}\sum_{j=2}^{n}u_j^2+\beta\int_0^x(i\mu+k)e^{-2\int_0^x k(t)dt}dx\right\}$$

for some constant vectors η, c_j, β.

We choose the initial conditions at the origin $0\in\mathbb{C}_1^{n+1}$ such that

$$\psi(0)=(1/\sqrt{-c},0,\dots,0),$$
$$\psi_x(0)=(0,1,0,\dots,0),\psi_{u_2}(0)=(0,0,1,\dots,0),\dots,\psi_{u_n}(0)=(0,\dots,0,1).$$

Then we obtain (9.4) which completes the proof of Theorem 9.1. ∎

Remark 9.2: The Lagrangian H-umbilical submanifolds defined by (9.4) in Statement (ii-3) of Theorem 9.1 are neither in the form of (9.2) nor in the form of (9.3), in general. For example, if we choose $c = -1$ and $\mu(x) = \sin x$. Then

$$k = \cos x, \lambda = 1 + 2\sin x, g = dx^2 + e^{2\sin x}(du_2^2 + \cdots + du_n^2).$$

So, by applying (9.4) we obtain

$$\psi(x, u_2, \dots, u_n) = e^{-ie^{ix}} \Bigg(1 + \sum_{j=2}^{n} \frac{u_j^2}{2} - \int_0^x e^{-2\sin x + ix} dx,$$

$$- \sum_{j=2}^{n} \frac{y^2}{2} + \int_0^x e^{-2\sin x + ix} dx, u_2, \dots, u_n \Bigg), \tag{9.27}$$

which is neither in the form of (9.2) nor in the form of (9.3).

10. Lagrangian H-umbilical surfaces in $\mathbb{C}H^2$

Assume M is a Lagrangian H-umbilical surface in $\mathbb{C}H^2$ with

$$h(e_1, e_1) = \lambda J e_1, \quad h(e_1, e_2) = \mu J e_2, \quad h(e_2, e_2) = \mu J e_1, \tag{10.1}$$

for some functions λ, μ with respect to some orthonormal local frame fields.

Put

$$k = \frac{\mu'}{\lambda - 2\mu}.$$

THEOREM 10.1:

(i) *For each constant $\hat{c}$, there exists a non-totally geodesic Lagrangian H-umbilical isometric immersion ℓ: $M(\hat{c}) \to \mathbb{C}H^2(4c)$ of a surface $M(\hat{c})$ of constant curvature $\hat{c}$ into $\mathbb{C}H^2(4c)$ such that the integral curves of JH are geodesics in $M(\hat{c})$.*

(ii) *Let $\bar{\psi}$: $M \to \mathbb{C}H^2(4c)$ be a Lagrangian H-umbilical isometric immersion of a surface into $\mathbb{C}H^2(4c)$ satisfying (10.1). If M contains no open subsets of constant sectional curvature $\geq c$ and if integral curves of JH are geodesics in M, then*

(ii-1) *if $u(x) = c + \mu^2(x) + k^2(x) > 0$, there exists a unit speed Legendre curve $z = (z_1, z_2)$: $I \to H_1^3(c) \subset \mathbb{C}_1^2$ such that up to rigid motions of $\mathbb{C}H^2(4c)$, $\bar{\psi}$ is locally given by $\pi \circ \psi$ where*

$$\psi(x, y) = (z_1(x), z_2(x)\cos y, z_2(x)\sin y); \tag{10.2}$$

(ii-2) *if* $u = c + \mu^2 + k^2 < 0$, *then there exists a unit speed Legendre curve* $z = (z_1, z_2)\colon I \to H_1^3(c) \subset \mathbb{C}_1^2$ *such that up to rigid motions of* $\mathbb{C}H^2(4c)$, $\bar{\psi}$ *is locally given by* $\pi \circ \psi$ *where*

$$\psi(x, y) = (z_1(x) \cosh y, z_1(x) \sinh y, z_2(x)); \tag{10.3}$$

(ii-3) *if* $u = c + \mu^2 + k^2 = 0$, *then up to rigid motions of* $\mathbb{C}H^2(4c)$, $\bar{\psi}$ *is locally given by* $\pi \circ \psi$ *where*

$$\begin{aligned}\psi(x, y) = e^{\int_0^x (i\mu + k)dx} \Bigg(\frac{1}{\sqrt{-c}} \left(1 - \frac{cy^2}{2} - \int_0^x (i\mu + k) e^{-\int_0^x 2k(t)dt} dx\right), \\ (i\mu(0) - k(0)) \left(\frac{y^2}{2} + \frac{1}{c} \int_0^x (i\mu + k) e^{-\int_0^x 2k(t)dt} dx\right), y\Bigg).\end{aligned} \tag{10.4}$$

Proof: We divide the proof of Statement (i) into two cases.

CASE (i-1): $\hat{c} > c$. (6.2-1), (6.2-2) and (6.2-3) give the Lagrangian H-umbilical isometric immersion of $M(\hat{c})$ into $\mathbb{C}H^2(4c)$ such that integral curves of JH are geodesics.

CASE (i-2): $\hat{c} \leq c$. Put $\hat{c} = -\delta^2$. Let M be the warped product surface $M = \mathbb{R} \times_{\frac{1}{\delta}e^{\delta x}} \mathbb{R}$. Then M is of constant curvature $\hat{c} < 0$. Put

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \delta e^{-\delta x} \frac{\partial}{\partial y}. \tag{10.5}$$

Then e_1, e_2 form an orthonormal frame field such that $\omega_2^1 = -\delta\omega^2$.

Consider the following first order ordinary differential equation

$$\lambda'(x) = -\frac{2\delta(\lambda^2 + 4c + 4\delta^2)}{\sqrt{\lambda^2 + 4c + 4\delta^2} + \lambda}. \tag{10.6}$$

Choose an initial condition $\lambda(0) = \lambda_0$ with $\lambda_0 > 0$. Since $c > 0$, (10.5) together with the initial condition has a unique solution $\lambda = \lambda(x)$ on an open neighborhood, say $\hat{I}$, of 0. By using the solution λ, we define a function μ by

$$\mu(x) = \frac{1}{2}(\lambda + \sqrt{\lambda^2 + 4c + 4\delta^2}). \tag{10.7}$$

We define a symmetric bilinear form σ on $M(\hat{c})$ by

$$\sigma(e_1, e_1) = \lambda e_1, \quad \sigma(e_1, e_2) = \mu e_2, \quad \sigma(e_2, e_2) = \mu e_1. \tag{10.8}$$

(10.5)–(10.8) implies that $M(\hat{c})$ together with σ satisfies the three conditions given in Theorem A. Therefore, there exists a Lagrangian H-umbilical isometric immersion ℓ: $M(\hat{c}) \to \mathbb{C}H^2(4c)$ with $h = J\sigma$ as its second fundamental form. It is easy to verify that the integral curves of JH are geodesics.

Statement (ii) can be proved as Statement (ii) of Theorem 9.1 with minor modifications. ∎

Remark 10.2: As Theorem 8.1, the assumption on the integral curves of JH made in statement (ii) of Theorem 10.1 cannot be omitted.

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